Topology Project

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I had no time to prove *Lemma 2*. The general *Borsuk-Ulam* theorem requires advanced concepts in Algebraic Topology. The part of complex analysis and winding numbers is complete but needs more writing.

Preliminaries

Graph Theory

Definition. A coloring is proper if no adjacent vertices are assigned the same color.

Definition. The chromatic number $\chi(G)$ is the smallest number of colours to colour a graph.

Complex Analysis

Definition. A curve γ in open $U \subseteq \mathbb{C}$ is

$$\gamma : [a, b] \to U$$

Notation. The curve γ could be expressed as a parameteric function $f(e^{2\pi it})$ for $0 \leq t \leq 1$.

Definition. The continuity of a complex-valued function is defined in an analogous manner to real-valued functions.

Definition. For continuous $F : [a, b] \to \mathbb{C}$, the integral of $[a, b]$ is

$$\int_a^b F(t)\ dt = \int_a^b u(t)\ dt + i \int_a^b v(t)\ dt$$

Definition. The winding number with respect to point α over closed path γ is

$$W(\gamma, \alpha) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - \alpha}\ dz$$

Definition. The degree of a function $deg(f)$ is the wind number $W(f(e^{2\pi it}))$.

Other

Fact. Vectors of d-dim sphere are exactly the vectors of the (d+1)-dim Euclidean space whose norm is 1.

Fact. The equator of a d-dim sphere is a subspace of the d-dim Euclidean space.

Definition. The distance between a point x and a closed set C_i is $dist(x, C_i) = \inf_{y \in C_i} ||y - x||$, i.e the distance between x and the closest point of C_i.

Fact. If point p is in closed set C_i, then $dist(p, C_i) = 0$, and if not then $dist(p, C_i) > 0$.

Definition. Two points of a sphere are antipodal if they are diametrically opposite, i.e expressed as p and $-p$.

Definition. The open hemisphere of pole x is $H(x) = \{y \in \mathbb{S}^d \mid \langle x, y \rangle > 0\}$.

Topological Methods

Lemma 1. Given $f : S^1 \to S^1$, and $f(-z) = -f(z)\ \forall z \in S^1$, then $deg(f)$ is odd.

Proof. Recall by definition $deg(f) = wind(f(e^{2\pi it}))$. Set $g : [0, 1] \to S^1$ by $g(t) = f(e^{2\pi it})$.

If $z = e^{2\pi it}$ then $-z = e^{\pi i} = e^{\pi i}z = e^{\pi i}e^{2\pi it} = e^{\pi i + 2\pi it} = e^{2\pi i(t+1/2)}$.

If $0 \leq t \leq \frac{1}{2}$ then $\frac{1}{2} \leq t + \frac{1}{2} \leq 1$. We have $g(t + \frac{1}{2}) = f(e^{2\pi i(t+1/2)}) = f(-z) = -f(z) = -f(e^{2\pi it}) = -g(t)$.

For a partition of $[0, 1]$ into $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$, partition $[0, \frac{1}{2}]$ into $\bigcup_{k=1}^{n} [t_{k-1}, t_k]$, where $|\theta_k| < \frac{\pi}{2}$ for $\frac{g(t_k)}{g(t_{k-1})} = e^{i\theta_k}$. Partition $[\frac{1}{2}, 1] = \bigcup_{k=1}^{n} [\frac{1}{2} + t_{k-1}, \frac{1}{2} + t_k]$.

It follows $\frac{g(\frac{1}{2}+t_k)}{g(\frac{1}{2}+t_{k-1})} = \frac{-g(t_k)}{-g(t_{k-1})} = e^{i\theta_k}$.

Observe an approximation of the integral of the winding number would be

$$\frac{1}{2\pi}\left(\sum_{i=1}^{n} f(z_i) \cdot \Delta_i\right) = \frac{1}{2\pi}(\theta_1 + \cdots + \theta_n + \theta_1 + \cdots + \theta_n)$$

Observe $g(\frac{1}{2}) = -g(0)$, and

$$\begin{aligned} \frac{g(t_n)}{g(t_{n-1})} \cdot \frac{g(t_{n-1})}{g(t_{n-2})} \cdot \cdots \cdot \frac{g(t_2)}{g(t_1)} \cdot \frac{g(t_1)}{g(t_0)} &= \frac{t_n}{t_0} = \frac{1/2}{0} = -1 \\ e^{i\theta_n} \cdot e^{i\theta_{n-1}} \cdot \cdots \cdot e^{i\theta_2} \cdot e^{i\theta_1} &= \\ e^{i(\theta_1+\theta_2+\cdots+\theta_n)} &= \end{aligned}$$

But we know $e^{i\pi} = -1$. Hence $\theta_1 + \theta_2 + \cdots + \theta_n = \pi + 2\pi N$ for some integer N.

It follows the wind approximation is

$$\frac{1}{2\pi}((\pi + 2\pi N) + (\pi + 2\pi N)) = \frac{1}{2\pi}(2\pi + 4\pi N) = 1 + 2N$$

which is an odd number.

Indeed, a sequence of odd numbers converges to an odd number. Therefore, the winding number is odd.

Lemma 2. No map $f : S^2 \to S^1$ such that $f(-p) = -f(p)\ \forall p \in S^2$.

Theorem. 2-dim Borsuk-Ulam. If $f : S^2 \to \mathbb{R}^2$ is continuous, then $\exists p \in S^2$ such that $f(-p) = f(p)$.

Proof. Suppose towards contradiction $f(x) \neq f(-x)\ \forall x \in S^2$. Construct map $g : S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||_2} \in S^1$$

Observe

$$\forall x \in S^2 \quad g(-x) = \frac{f(-x) - f(-(-x))}{||f(-x) - f(-(-x))||_2} = \frac{f(-x) - f(x)}{||f(-x) - f(x)||_2} = -g(x)$$

Contradicting *lemma 2.*

Theorem. Borsuk-Ulam. If $f : S^n \to \mathbb{R}^n$ is continuous, then $\exists p \in S^n$ such that $f(-p) = f(p)$.

The proof is a natural generalization but requires concepts beyond a first course in topology.

Theorem. Lyusternik & Shnirel'man. If S^n is covered by closed sets $C_1, C_2, \ldots, C_n, C_{n+1}$, then there $p \in S^n$ and C_i such that $p, -p \in C_i$.

Proof. Assume towards contradiction that if $p \in C_i$ then $-p \notin C_i$ for all $p \in S^n$. Define functions $f_1, f_2, \ldots, f_{n+1} : S^n \to \mathbb{R}$ by $f_i(x) = dist(x, C_i)$. Construct $f : S^n \to \mathbb{R}^n$ by $f(x) = (f_1(x) - f_{n+1}(x), f_2(x) - f_{n+1}(x), \ldots, f_n(x) - f_{n+1})$.

By *Borsuk-Ulam* there are $p, -p \in S^n$ such that $f(p) = f(-p)$. It follows for all $1 \leq i, j \leq n+1$

$$f_i(p) - f_{n+1}(p) = f_i(-p) - f_{n+1}(-p)$$
$$f_j(p) - f_{n+1}(p) = f_j(-p) - f_{n+1}(-p)$$

So $f_i(p) - f_j(p) = f_i(-p) - f_j(-p)$.

Since $p \in S^n = C_1 \cup \cdots \cup C_{n+1}$, we get $p \in C_i$ for some i. Similarly $-p \in S^n$ so $-p \in C_j$ for some j. By hypothesis $i \neq j$. Clearly $f_i(p) = f_j(-p) = 0$. Then

$$f_i(p) - f_j(p) = f_i(-p) - f_j(-p)$$
$$-f_j(p) = f_i(-p)$$

But $f_j(p) > 0$ since $p \notin C_j$, and similarly $f_i(p) > 0$. So we have an equality between a negative and a positive number. Contradiction.

Theorems

Definition. The Kneser graph $KG_{n,k}$ for $n \geq 2,\ k \geq 1$, has vertex set $C([n], k)$, and any two vertices $u, v \in C([n], k)$ are adjacent if and only if they are disjoint, i.e. $u \cap v = \phi$.

Theorem. The chromatic number of the Kneser graph $KG_{n,k}$ is $n - 2k + 2$.

Fix n and k. Assume for the sake of contradiction, the chromatic number of Kneser graph $KG_{n,k}$ is less than $n - 2k + 2$. Then we have a proper coloring $c : C([n], k) \to \{1, \ldots, n-2k+1\}$ using at most $n-2k+1$ colors. Set $d = n-2k+1$ and take a set X of n vectors on the d-dim sphere $\mathbb{S}^d$ where any $d+1$ vectors are linearly independent.

Let $U_i = \{x \in \mathbb{S}^d \mid \exists k\text{-set } S \subset X, c(S) = i, S \subset H(x)\}$ for $i = 1, \ldots, d$, and take complement $A = \mathbb{S}^d \setminus (U_1 \cup \cdots \cup U_d)$. We claim each U_i is open.

To see why, fix a point $y \in S^d$, and observe $U_y = \{x \in S^{n-1} : \langle x, y\rangle > 0\}$ is open as it is the preimage of the open set $(0, \infty)$ under the continuous map $f_y(x) = \langle x, y\rangle$.

For finite k-subset $B = \{y_1, \ldots, y_k\}$, Observe

$$U_B = \bigcap_{j=1}^{k} U_{y_j} = \left\{x \in S^{n-1} : \langle x, y_j\rangle > 0\ \forall j\right\}$$

is an intersection of finitely many open sets, hence *open.*

Therefore $U_i = \bigcup_{\substack{B \in \binom{[n]}{k} \\ c(B)=i}} U_B$ is a union of open sets, hence *open.* Moreover complement A is closed.

Clearly A alongside U_i do cover $\mathbb{S}^d$. So if none of them contains a pair of antipodal points, then neither does $\mathbb{S}^d$, hence contradicting the *Lyusternik & Shnirel'man* theorem. We aim to reach that contradiction. Consider $x \in \mathbb{S}^d$.

Case 1. $x \in U_i$, i.e $H(x)$ contains a k-subset colored with color i, corresponding to a vertex colored i. Since $H(x)$ and $H(-x)$ are disjoint, any k-subset in $H(-x)$, is disjoint from any k-subset in $H(x)$. Thereby, corresponding vertices are adjacent. Since the coloring is proper by hypothesis, $H(-x)$ does not contain a k-subset colored with i, hence $-x \notin U_i$.

Case 2. $\pm x \in A$. By definition of A, neither $H(x)$ nor $H(-x)$ contains a k-subset of X. Hence each of $H(x)$ and $H(-x)$ contains at most $k-1$ vectors. It follows there is at least $n - 2(k-1) = n - 2k + 2 = d+1$ points in the equator $\{y \in \mathbb{S}^d \mid \langle x, y\rangle = 0\}$, contained in a subspace of dim d, concluding they are linearly dependent. Contradiction.

Finally we show a valid constructive coloring of $KG_{n,k}$ using $n - 2k + 2$ colors. Color each k-set with all elements in $[2k-1]$ with one color, and every other k-set by their largest element. Thereby we use at most $n-(2k-1)+1=n-2k+2$ colors, where all k-sets of a given color intersect.

Resources

- Borsuk-Ulam Theorem by Nicolas Bourbaki
- Kneser's Conjecture by Nicolas Bourbaki
- A Course in Topological Combinatorics, Chapter 2, by Longueville
- Lecture 14. Combinatorics by Jacob Fox