Kneser Graph

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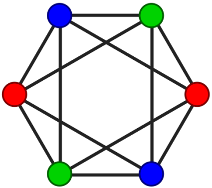
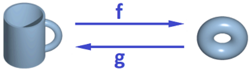
Discrete & Continuous

Graph Chromatic Number



Continuous Functions over Topological Spaces

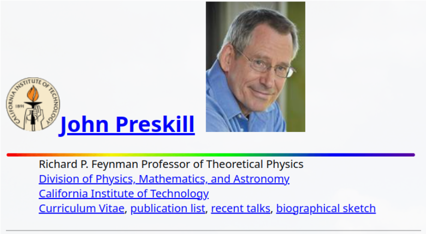


Functional Analysis from Physics

- Early 20th-century physicists worked with differential equations to model quantum phenomena.
- Functional analysis emerged for a rigorous formulation.

Physicists and Computer Science

Geoffry Hinton wins the Nobel Prize



Caltech R. Feynman on Quantum Computation and Information

CS Theory and Computing

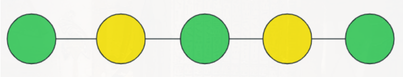
Combinatorics and Algorithm Design, 1960s

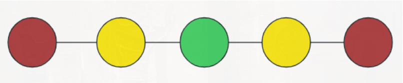
Scientists, Engineers, and Entrepreneurs, Nowadays

Preliminary: Graph Theory

Proper Coloring. Assignment of colors to vertices, whereby adjacent vertices are colored differently.

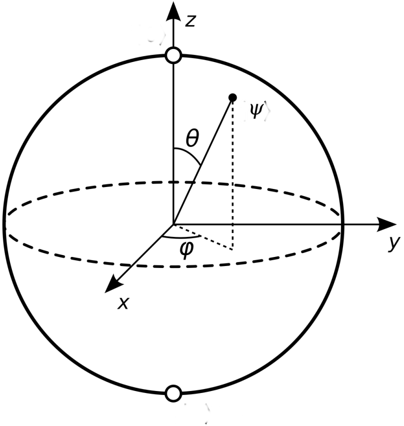
Chromatic Number. The smallest number of colors, allowing a proper coloring.

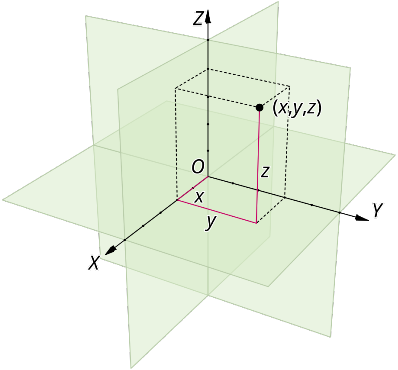




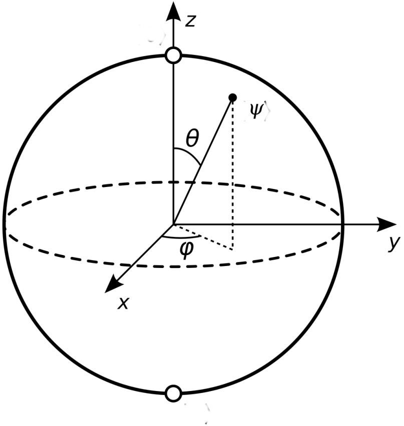
Preliminary: Linear Algebra

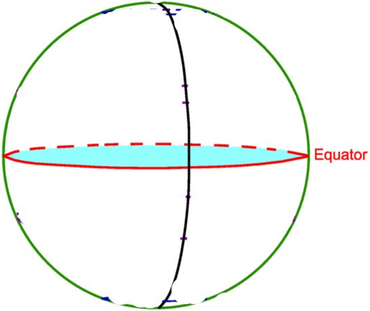
Fact. Vectors of d-dim sphere are exactly the vectors of the (d+1)-dim Euclidean space whose norm is 1.



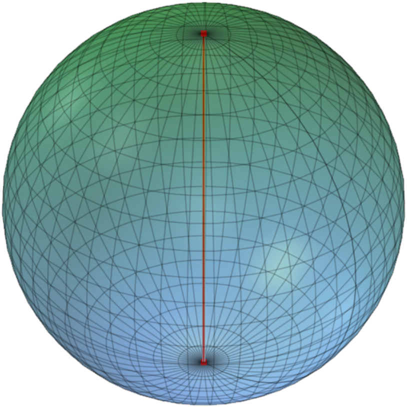


Fact. The equator of a d-dim sphere is a subspace of the d-dim Euclidean space.

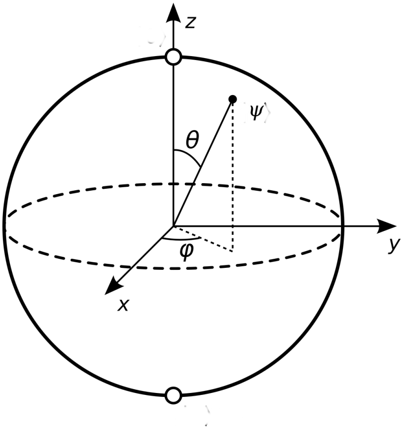


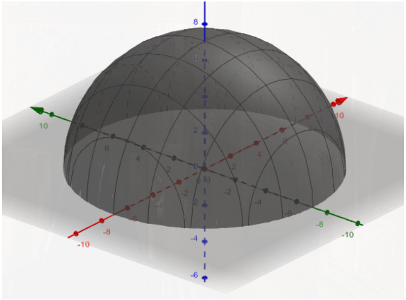


Definition. Two points of a sphere are antipodal if they are diametrically opposite, i.e expressed as p and $-p$.



Definition. The open hemisphere of pole x is $H(x) = \{y \in \mathbb{S}^d \mid \langle x, y \rangle > 0\}$.





Topological Methods

Theorem. Borsuk-Ulam. If $f : S^n \to \mathbb{R}^n$ is continuous, then $\exists p \in S^n$ such that $f(-p) = f(p)$.

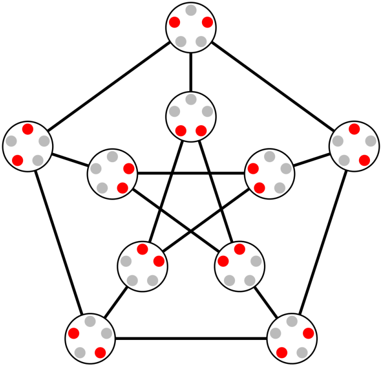
Note. The 2-dim case of Borsuk-Ulam is easier to show.

Corollary. Lyusternik & Shnirel'man. If S^n is covered by open or closed sets $C_1, C_2, \ldots, C_n, C_{n+1}$, then there $p \in S^n$ and C_i such that $p, -p \in C_i$.

Note. In some contexts called a variant of *Borsuk-Ulam*.

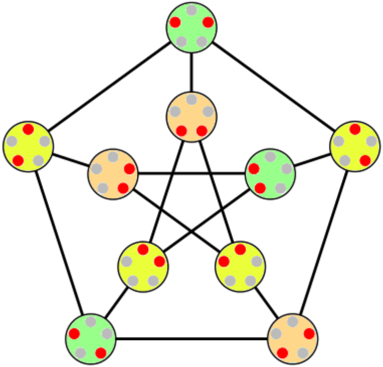
Kneser Graph

Definition. The Kneser graph $KG_{n,k}$ for $n \geq 2$, $k \geq 1$, has vertex set $C([n], k)$, and any two vertices $u, v \in C([n], k)$ are adjacent if and only if they are disjoint, i.e. $u \cap v = \phi$.



Lovász

Theorem. The chromatic number of the Kneser graph $KG_{n,k}$ is $n - 2k + 2$.



Proof: Not less than $n-2k+2$

Fix n and k. Assume for the sake of contradiction, the chromatic number of Kneser graph $KG_{n,k}$ is less than $n-2k+2$. Then we have a proper coloring $c: C([n], k) \to \{1, \ldots, n-2k+1\}$ using at most $n-2k+1$ colors.

Set $d = n-2k+1$ and take a set X of n vectors on the d-dim sphere $\mathbb{S}^d$ where any $d+1$ vectors are linearly independent.

Let $U_i = \{x \in \mathbb{S}^d \mid \exists k\text{-set } S \subset X, c(S) = i, S \subset H(x)\}$ for $i = 1, \ldots, d$, and take complement $A = \mathbb{S}^d \setminus (U_1 \cup \cdots \cup U_d)$.

Each U_i is open.

To see why, fix a point $y \in S^d$, and observe $U_y = \{x \in S^{n-1} : \langle x, y \rangle > 0\}$ is open as it is the preimage of the open set $(0, \infty)$ under the continuous map $f_y(x) = \langle x, y \rangle$.

For finite k-subset $B = \{y_1, \dots, y_k\}$, Observe

$$U_B = \bigcap_{j=1}^{k} U_{y_j} = \left\{x \in S^{n-1} : \langle x, y_j \rangle > 0 \ \forall j\right\}$$

is an intersection of finitely many open sets, hence *open*. Therefore $U_i = \bigcup_{\substack{B \in \binom{[n]}{k} \\ c(B) = i}} U_B$ is a union of open sets, hence *open*. Moreover complement A is closed.

Clearly A alongside U_i do cover $\mathbb{S}^d$. So if none of them contains a pair of antipodal points, then neither does $\mathbb{S}^d$, hence contradicting the *Lyusternik & Shnirel'man* theorem. We aim to reach that contradiction.

Consider $x \in \mathbb{S}^d$.

Case 1. $x \in U_i$, i.e $H(x)$ contains a k-subset colored with color i, corresponding to a vertex colored i. Since $H(x)$ and $H(-x)$ are disjoint, any k-subset in $H(-x)$, is disjoint from any k-subset in $H(x)$. Thereby, corresponding vertices are adjacent. Since the coloring is proper by hypothesis, $H(-x)$ does not contain a k-subset colored with i, hence $-x \notin U_i$.

Case 2. $\pm x \in A$. By definition of A, neither $H(x)$ nor $H(-x)$ contains a k-subset of X. Recall by our construction, every k-subset is assigned a color. Hence each of $H(x)$ and $H(-x)$ contains at most $k-1$ vectors. It follows there is at least $n-2(k-1)=n-2k+2=d+1$ points in the equator $\{y \in \mathbb{S}^d \mid \langle x, y\rangle = 0\}$, contained in a subspace of dim d, concluding they are linearly dependent. Contradiction.

Proof: $n - 2k + 2$ coloring

We show a valid constructive coloring of $KG_{n,k}$ using $n - 2k + 2$ colors. Color each k-set with all elements in $[2k - 1]$ with one color, and every other k-set by their largest element. Thereby we use at most $n - (2k - 1) + 1 = n - 2k + 2$ colors, where all k-sets of a given color intersect.