Homework 1

Mostafa Touny

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Contents

Exercises

Sections 12 & 13, pages 83-84.

1

For every $x \in A$, we know there exists an open U_x such that $x \in U_x \subset A$. Observe $A = \cup_{x \in A} U_x$ is expressed as unions of open sets. By *axiom 2*, A is open. ■

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Lemma. $\mathcal{T}_c$ is a topology.

As $X - \phi = X$ and $X - X = \phi$, we get $\phi, X \in \mathcal{T}$.

For a collection of open sets $U = \bigcup_{x \in I} U_x$:

- Case. $U_x = \phi$ for all $x \in I$. Then clearly $U = \phi$ is open.
- Case. $U_{x_0} \neq \phi$ for some $x_0 \in I$. Then $X - U = X - \bigcup_{x \in I} = \bigcap_{x \in I} (X - U_x) \subset X - U_{x_0}$. It follows $X - U$ is countable as it is a subset of a countable set.

For open U_0 and U_1, if either is empty then $U_0 \cap U_1 = \phi$ is open. If both $X - U_0$ and $X - U_1$ are countable then so is $(X - U_0) \cup (X - U_1) = X - (U_0 \cap U_1)$.

Lemma. $\mathcal{T}_\infty$ is not a topology in general.

We show that by a counter-example. Let $X = \mathcal{R}$, $U_0 = R - \{0, 2, 4, \dots\}$, and $U_1 = R - \{0, 1, 3, \dots\}$. Clearly $R - U_0$ and $R - U_1$ are both infinite but $R - (U_0 \cup U_1) = R - (R - \{0\}) = \{0\}$ is non-empty finite, and hence not open.

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a.

$\bigcap \mathcal{T}_\alpha$ is a topology by lifting every property common along all T_α to $\bigcap \mathcal{T}_\alpha$.

$\bigcup \mathcal{T}_\alpha$ is not a topology in general. Consider $X = \{a, b\}$, $\mathcal{T}_a = \{\phi, \{a\}, X\}$, and $\mathcal{T}_b = \{\phi, \{b\}, X\}$. Observe $\mathcal{T}_a \cup \mathcal{T}_b = \{\phi, \{a\}, \{b\}, X\}$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_a \cup \mathcal{T}_b$.

b.

Lemma. Unique smallest.

Let F be the family of all topologies containing $\{\mathcal{T}_\alpha\}$. F is non-empty as the discrete topology $\mathcal{T}_{\text{disc}}$ is finer than any topology on X. By *(a)*, $\bigcap F$ is a topology. By minimality in set theory, it is smallest and unique.

Lemma. Unique greatest.

Observe $\mathcal{T} = \bigcap_\alpha \mathcal{T}_\alpha$ is a topology by *(a)*. It is largest as any open set U common in all topologies will be in $\mathcal{T}$. For any largest such topology $\mathcal{T}'$, by definition $\mathcal{T}' \subset \bigcap_\alpha \mathcal{T}_\alpha$ and $|\mathcal{T}'| \geq \mathcal{T}$, implying $\mathcal{T}' = \mathcal{T}$.

c.

Smallest is $\{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{b\}\}$, and largest is $\{\phi, X, \{a\}\}$.

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Lemma. $\mathcal{R}_l$ is not finer than $\mathcal{R}_k$.

Consider $0 \in B = (-1, 1) - K$ for $\mathcal{T}''$ in $\mathcal{R}_k$. In $\mathcal{R}_l$, for any $[y, x)$ where $y \leq 0 < x$, there is a small enough $1/n$, concluding $[y, x) \not\subseteq B$.

Lemma. $\mathcal{R}_k$ is not finer than $\mathcal{R}_l$.

Consider $0 \in [0, 1)$ in $\mathcal{R}_l$. For any (x', y') containing 0, it follows $x' < 0 < y'$, and hence $(x', y') \not\subseteq [0, 1)$.

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$\mathcal{T}_1$ contains $\mathcal{T}_4$.

$\mathcal{T}_2$ contains $\mathcal{T}_1$.

$\mathcal{T}_5$ contains $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_4$.

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a.

Since $\mathcal{Q} \subseteq \mathcal{R}$, it follows $\mathcal{B} \subseteq \mathcal{T}$. Take arbitrary open (a, b) with x contained in it. By density of rationals, there are $q, p \in \mathcal{Q}$ such that $a < q < x < p < b$. Hence $x \in (q, p) \subseteq (a, b)$.

b.

Since a basis generates a unique topology, if $\mathcal{T} = \mathcal{T}'$ then $\mathcal{T}$ and $\mathcal{T}'$ have common bases (plural of basis). It suffices to show $\mathcal{C}$ is not a basis of the *lower limit topology*.

We show the generated topology $\mathcal{T}$ by $\mathcal{C}$ is missing an element in $\mathcal{R}_l$. Take irrational $a_0 \in [a_0, b_0) \in \mathcal{R}_l$. For any basis element $B = [q, p) \in \mathcal{C}$, either $q > a_0$ or $q < a_0$. For the former, $a_0 \notin B$. For the latter $B \not\subseteq [a_0, b_0)$. Hence $[a_0, b_0) \notin \mathcal{T}$.