Homework 04

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Contents

Exercises

Section 18, pages 111-112.

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Yes. Take arbitrary open $U \ni f(x)$. By continuity $f^{-1}(U)$ is open. Moreover $x \in f^{-1}(U)$. By hypothesis $f^{-1}(U) \cap A \neq \phi$. It follows $\phi \neq f\left(f^{-1}(U) \cap A\right) \subseteq f(f^{-1}(U)) \cap f(A) = U \cap f(A)$.

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(a)

($\leftarrow$) For open $U \subseteq X$, by hypothesis $i^{-1}(U) = U$ is also open in X'.

($\rightarrow$) Take $U \in \mathcal{T}$. By definition $U \subseteq X$. By hypothesis $i^{-1}(U) = U$ is open in X', i.e $U \in \mathcal{T}'$.

(b)

By (a), i continuous $\longleftrightarrow \mathcal{T}' \supseteq \mathcal{T}$.

Consider the continuous $i^{-1} : X \to X'$ map. By (a), i^{-1} continuous $\longleftrightarrow \mathcal{T}' \subseteq \mathcal{T}$.

The intended conclusion follows.

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Construct $f : (a, b) \to (0, 1)$ where $f(x) \mapsto \dfrac{x-a}{b-a}$. It is injective as $\dfrac{x-a}{b-a} = \dfrac{x'-a}{b-a}$ implies $x = x'$, and surjective as for $y \in (0, 1)$ we can take x such that $b > x = y(b-a) + a > a$ implying $f(x) = y$. Hence f is bijective.

Observe for $(c, d) \subseteq (0, 1)$ we have $f^{-1}(c, d) = (c(b-a)+a, d(b-a)+a)$. For an arbitrary open $U \subseteq (0, 1)$, we know $U = \bigcup_n (a_n, b_n)$. Thereby $f^{-1}(U_n(a_n, b_n)) = \bigcup_n f^{-1}(a_n, b_n)$ a union of open sets, which in turn is open.

To show $[a, b]$ is homeomorphic with $[0, 1]$, consider the function $f : [a, b] \to [0, 1]$ where $f(x) \mapsto \dfrac{x-a}{b-a}$. Then for $y \in [0, 1]$ we can take x such that $b \geq x = y(b-a) + a \geq a$. The remaining parts of the proof are analogous.

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(a)

Observe given a well-defined $f : X \to Y$, if some $x \in A \cap B$ then $f|A(x) = f|B(x)$. Hence, the *Pasting Lemma* is applicable.

It follows if $f|(A_1 \cup \cdots \cup A_{N-1})$ is continuous and $f|A_N$ is continuous, then so is $f|(A_1 \cup \cdots \cup A_N)$. By ordinary induction the intended result follows on any finite collection $\{A_\alpha\}$.

(b)

Lemma. Let Y be a subspace of X. if U is closed in X and $U \subseteq Y$, then U is closed in Y.

We know $X - U$ is open in X. Then $Y \cap (X - U)$ is open in Y. It follows

$$\begin{aligned} Y \cap (X - U) &= (Y \cap X) \cap (Y - U) \\ &= Y \cap (Y - U) \\ &= Y - U \end{aligned}$$

Thus $Y - (Y - U) = U$ is closed in Y.

Theorem. main problem.

Consider the function $f : (0, 1) \to \mathbb{R}$ where $x \mapsto 2x$. It is not continuous as $[0, 1]$ is closed in $\mathbb{R}$ but $f^{-1}([0, 1]) = (0, 1/2]$ is not closed in $(0, 1)$.

Take $A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ and observe $\bigcup_n^\infty A_n = (0, 1)$.

Let B be an arbitrary closed set in $\mathbb{R}$. Then $\{y/2 \mid y \in B\}$ is closed in $\mathbb{R}$. To see why, take z a limit point of it. Then $2z$ would be a limit point of B and it follows $2z \in B$, concluding $2z/2 = z$ is contained in the set.

Thereby $f^{-1}|A_n(B) = \{y/2 \mid y \in B\} \cap A_n$ is closed in $\mathbb{R}$. Since A_n is a subspace of $\mathbb{R}$ and $f^{-1}|A_n(B) \subseteq A_n$, by our lemma, we conclude $f^{-1}|A_n(B)$ is closed in A_n also.

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Let $U \times V$ be an arbitrary open set of $B \times D$. By definition U and V are respectively open in B and D. By hypothesis the following are open sets

$$\begin{aligned} f^{-1}(U) &= \{a \in A \mid f(a) \in U\} \\ g^{-1}(V) &= \{c \in C \mid g(c) \in V\} \end{aligned}$$

Moreover, by definition

$$\begin{aligned}(f \times g)^{-1}(U \times V) &= \{(a, c) \in A \times C \mid f(a) \in U \land f(c) \in V\} \\ &= f^{-1}(U) \times g^{-1}(V)\end{aligned}$$

Which is open by definition of product topology.

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Let $g_1 : \bar{A} \to Y$ and $g_2 : \bar{A} \to Y$ be two extensions of f. Then $g_1(x) = g_2(x)\ \forall x \in A$ (1). Take $x \in \bar{A}$ and assume towards contradiction $g_1(x) \neq g_2(x)$.

Note $\forall U \ni x$ open, $U \cap A \neq \phi$ (2).

Since Y is Hausdorff, there are open sets $V_1 \ni g_1(x)$ and $V_2 \ni g_2(x)$ where $V_1 \cap V_2 = \phi$ (3).

By continuity of g_1 and g_2 along *thm 18.1*, there are open $U_1 \ni x$ and $U_2 \ni x$ such that $g_1(U_1) \subseteq V_1$ and $g_2(U_2) \subseteq V_2$ (4).

Take open $U = U_1 \cap U_2$ and note $U \ni x$, implying by (2) $\exists x_0 \in U \cap A$. By (1), $g_1(x_0) = g_2(x_0)$. By (4), a contradiction of (3) is reached ■