Homework 05

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May 26, 2025

Contents

Exercises

Section 19

1

We satisfy the definition of a basis for a *Box Topology*.

For $\mathbf{x} = (x_\alpha) \in \prod_\alpha X_\alpha$ we have $x_\alpha \in X_\alpha$, implying the existence of a basis element B_α in X_α, where $x_\alpha \in B_\alpha$. Hence $\mathbf{x} \in \prod_\alpha B_\alpha$.

Assume $\mathbf{x} \in \prod_\alpha B_\alpha \cap \prod_\alpha B'_\alpha$. Then $x_\alpha \in B_\alpha \cap B'_\alpha$, implying the existence of B''_α, where $x_\alpha \in B''_\alpha \subset B \cap B'$. Hence $\mathbf{x} \in \prod B''_\alpha \subset \prod B'_\alpha \cap \prod B''_\alpha$.

For a *Product Topology*, the proof is similar, except we will have finitely many B_α. Fix α and observe $x_\alpha \in X_\alpha \subset X_\alpha$ where X_α is open.

2

We set the following notation:

$\mathcal{B}_{A_\alpha}$	basis of A_α
$\mathcal{B}_{\Pi X_\alpha} = \{\Pi_\alpha B_\alpha \mid \text{finitely } B_\alpha \in \mathcal{B}_{X_\alpha}, \text{ and remaining } B_\alpha = X_\alpha\}$	basis of product topology $\Pi_\alpha X_\alpha$
$\mathcal{B}_{\Pi A_\alpha} = \{\Pi_\alpha B_\alpha \mid \text{finitely } B_\alpha \in \mathcal{B}_{A_\alpha}, \text{ and remaining } B_\alpha = A_\alpha\}$	basis of product topology $\Pi_\alpha A_\alpha$
$\mathcal{B}^{(s)}_{\Pi A_\alpha} = \{B \cap \Pi_\alpha A_\alpha \mid B \in \mathcal{B}_{\Pi X_\alpha}\}$	basis of the subspace induced by $\Pi_\alpha X_\alpha$

It suffices to show $\mathcal{B}_{\Pi A_\alpha} = \mathcal{B}^{(s)}_{\Pi A_\alpha}$.

Observe $X_\alpha \cap A_\alpha = A_\alpha$, and that for $B_\alpha \in \mathcal{B}_{X_\alpha}$ it follows $B_\alpha \cap A_\alpha$ is a basis element of $\mathcal{B}_{A_\alpha}$.

6

Take arbitrary $\mathbf{x} \in \prod X_\alpha$ and consider any neighbourhood U. Then we have a basis element B where $x \in B \subset U$. By definition $B = \prod U_\alpha$ where finitely many U_i are open in X_i for $i = 1, \ldots, k$, and remainings are exactly X_α. For each i, and by hypothesis, all but finitely many $x_n^{(i)}$ are in U_i. Let $U'_i = \{\mathbf{x}_n \mid x_n^{(i)} \in U_i\}$ and take $U' = \bigcap_{i=1}^k U'_i$. Observe all $(\mathbf{x}_n)$ except finite U' are in U. ■

Not true for box topology. As a counter example, from analysis we know $f_i(n) = \dfrac{i}{n}$ is point-wise convergent to 0 but not uniformly convergent to it. Accordingly set $x_n^{(i)} = \dfrac{i}{n}$ for product topology $\mathbb{R}^\omega = \prod_{n \in Z_+} \mathbb{R}$. ■

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We show R^∞ is closed to conclude $cl\,(R^\infty) = R^\infty$.

Let $\mathbf{x} = (x_1, x_2, \dots)$ be a limit point of R^∞. Then for each x_i, we can choose small enough $(a_i, b_i) \ni x_i$, to form an open $U = \prod_i (a_i, b_i)$ of R^ω. It follows, some $\mathbf{x}' = (x'_1, x'_2, \dots) \in U \cap R^\infty$. By definition, $\mathbf{x}'$ has some index k whereby $x'_j = 0$ for all $j \geq k$. But if $0 = x'_j \in (a_j, b_j)$ for arbitrarily small $(a_j, b_j) \ni x_j$, then necessarily $x_j = 0$. So for $\mathbf{x}$ we have $x_j = 0$ for all $j \geq k$, concluding $\mathbf{x} \in R^\infty$. $\blacksquare$

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(a)

Consider the set $\mathcal{S} = \bigcup_\alpha \{f_\alpha^{-1}(U_\alpha) \mid U_\alpha \text{ open in } X_\alpha\}$. The set of all topologies $\boldsymbol{\tau}_\beta$ containing $\mathcal{S}$ is non-empty as witnessed by the discrete topology. Taking $\bigcap_\beta \boldsymbol{\tau}_\beta$ is the unique coarsest topology containing $\mathcal{S}$.

The argument follows the same line of reasoning of *exercise 4* in *section 13*.

(b)

$\boldsymbol{\tau}_\mathcal{S} \supseteq \bigcap_\beta \boldsymbol{\tau}_\beta$

Generate a topology $\boldsymbol{\tau}_\mathcal{S}$ by $\mathcal{S}$ as a subbasis. Then by definition it contains all elements of $\mathcal{S}$.

$\boldsymbol{\tau}_\mathcal{S} \subseteq \bigcap_\beta \boldsymbol{\tau}_\beta$

Consider any topology $\boldsymbol{\tau}_\beta$ containing $\mathcal{S}$. Then by topology's axioms, $\boldsymbol{\tau}_\beta$ contains finite intersections of $\mathcal{S}$, and in turn arbitrary unions of those intersections. Hence $\boldsymbol{\tau}_\mathcal{S} \subseteq \boldsymbol{\tau}_\beta \, \forall \beta$. It follows $\boldsymbol{\tau}_\mathcal{S} \subseteq \bigcap_\beta \boldsymbol{\tau}_\beta$.

(c)

($\rightarrow$) Fix α. Take U_α open in X_α. Then $f_\alpha^{-1}(U_\alpha)$ is open in A relative to topology $\boldsymbol{\tau}$. By hypothesis $g^{-1}(f_\alpha^{-1}(U_\alpha)) = (f_\alpha \circ g)^{-1}(U_\alpha)$ is open in Y.

($\leftarrow$) Consider a basis element B in A. Relative to topology $\boldsymbol{\tau}$, we know $B = f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap f_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap f_{\alpha_k}^{-1}(U_{\alpha_k})$. For $i = 1, \dots, k$, since U_{α_i} is open in X_{α_i}, by hypothesis we have $(f_{\alpha_i} \circ g)^{-1}(U_{\alpha_i}) = g^{-1}(g_\alpha^{-1}(U_\alpha))$ is open.

By topology's axioms, $g^{-1}(f_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \cdots \cap g^{-1}(f_{\alpha_k}^{-1}(U_{\alpha_k}))$ is open. Since g is a well-defined function, uniquely assigning elements, g^{-1} is injective. It follows

$$g^{-1}(f_{\alpha_1}^{-1}(U_{\alpha_1})) \cap \cdots \cap g^{-1}(f_{\alpha_k}^{-1}(U_{\alpha_k})) = g^{-1}(f_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap f_{\alpha_k}^{-1}(U_{\alpha_k})) \quad \blacksquare$$

(d)

Proposition. If function $f : X \to Y$ maps each basis element B of X to a basis element B' of Y, then $f(U)$ is open in Y for every open U in X.

Lemma. For a fixed α, the image of $f_\alpha^{-1}(U_\alpha)$ in $\mathcal{T}$, is a basis element of $\mathcal{T}_Z$.

Observe the basis of $\mathcal{T}_Z$ is $\{\prod_\alpha U_\alpha \cap Z \mid \text{all are } X_\alpha \text{ except finitely } U_\alpha \text{ are open}\}$.

Fix α and consider $f_\alpha^{-1}(U_\alpha)$ in $\mathcal{T}$. Accordingly, consider $\prod_\beta U_\beta \cap Z$ where $U_\beta = X_\beta$ for $\beta \neq \alpha$. Note it is a basis element of $\mathcal{T}_Z$.

We claim $f(f_\alpha^{-1}(U_\alpha)) = \prod_\beta U_\beta \cap Z$.

($\rightarrow$) $f_\alpha(x) \in U_\alpha$ for $x \in f_\alpha^{-1}(U_\alpha)$.

($\leftarrow$) For arbitrary y in the R.H.S, it has at index α an element in U_α. So $y = f(x)$ where $x \in f^{-1}(U_\alpha)$.

Corollary. The image of a basis element of A is a basis element of $\mathcal{T}_Z$.

Following the same line of reasoning it can be shown a finite intersection $f_\alpha^{-1}(U_\alpha) \cap \cdots \cap f_\alpha^{-1}(U_\alpha)$ is a basis element of $\mathcal{T}_Z$.

Theorem. Main problem.

Follows by the *corollary* alongside the *proposition*.