Homework 06

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Contents

Exercises

Section 20

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(a)

By theorem *18.1 (4)* it suffices to take $d(x, x') \in (a, b)$ for $(x, x') \in X \times X$, and then construct a neighbourhood $U \times U'$ of (x, x') such that $d(U, U') \subset (a, b)$.

Observe if $\varepsilon = \varepsilon' \leq \dfrac{b - d(x, x')}{4}$, then taking $(x_0, x_1) \in B(x, \varepsilon) \times B(x', \varepsilon')$, yields

$$\begin{aligned} d(x_0, x_1) &\leq d(x_0, x) + d(x, x') + d(x', x_1) && \text{triangular inequality} \\ &\leq \frac{b - d(x, x')}{4} + \frac{b - d(x, x')}{4} + d(x, x') \\ &\leq \frac{b - d(x, x')}{2} + d(x, x') \\ &< b - d(x, x') + d(x, x') = b \end{aligned}$$

Similarly $\varepsilon = \varepsilon' \leq \dfrac{d(x, x') - a}{4}$ yields

$$\begin{aligned} d(x, x') &\leq d(x, x_0) + d(x_0, x_1) + d(x_1, x') && \text{triangular inequality} \\ &\leq \frac{d(x, x') - a}{2} + d(x_0, x_1) \\ d(x_0, x_1) &\geq d(x, x') - \frac{d(x, x') - a}{2} \\ &> d(x, x') - d(x_0, x_1) + a = a \end{aligned}$$

Taking the minimum values for ε and ε' concludes $d(B(x, \varepsilon) \times B(x', \varepsilon')) \subset (a, b)$.

(b)

4a

Consider $g(t) = (t, t, \dots)$ alongside continuity equivalence of *theorem 18.1 (4)*.

It is continuous in the product topology. For open neighbourhood V around $(t, t, \dots)$, all are X except finitely many open V_α. for $t \in (a_\alpha, b_\alpha)$, consider the distance $c_\alpha = \min\{t - a_\alpha, b_\alpha - t\}$. So we can take the minimum along these finite c_α and construct a neighbourhood U around t such that $f(U) \subset V$.

Not continuous in the box topology. A counter-example is $g(0) = (0, 0, \dots)$ with $V_\alpha = \left(\dfrac{-1}{n}, \dfrac{1}{n}\right)$. Taking any open $(a, b) \ni 0$ implies $\exists x > 0\ \forall n,\ x \in \left(\dfrac{-1}{n}, \dfrac{1}{n}\right)$. Contradiction.

Not continuous in the uniform topology. Consider $x \in \mathbb{R}^\omega$ such that $x_0 = 0$ and $x_\alpha \to 1/2$. Observe $f(0) = (0, 0, \dots) \in B(x, 1/2)$ as $\forall \alpha\ x_\alpha < 1/2$. Following the same line of reasoning of the preceeding case, we no open neighbourhood U of 0 satisfies $f(U) \subset B(x, 1/2)$.

4b

The sequence $(1, 1, \dots)$ is trivially convergent to 1 in all of product, box, and uniform topologies of $\mathbb{R}^\omega$.

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We characterize the set of limit points.

Lemma. A sequence $x = (x_1, x_2, \dots)$ whereby $x_i \not\to 0$ is not a limit point of $\mathbb{R}^\infty$.

By definition, there is a fixed ε_0, such that for each index α, there is some $i > \alpha$ where $|x_i - 0| > \varepsilon_0$. Consider neighbourhood $B\left(x, \frac{\varepsilon_0}{2}\right)$. It follows no element of $\mathbb{R}^\infty$ is in it.

Lemma. A sequence $x = (x_1, x_2, \dots)$ whereby $x_i \to 0$ is a limit point of $\mathbb{R}^\infty$.

For any neighbourhood $B(x, \varepsilon)$, by the convergence of x_i to 0, there is some N_0, such that $\forall j \geq N_0$, $0 \in (x_j - \varepsilon, x_j + \varepsilon)$. Consider the element x' whereby $x'_i = x_i$ for $i < N_0$ and $x'_i = 0$ for $i \geq N_0$. Observe x' is both in $\mathbb{R}^\infty$ and $B(x, \varepsilon)$.

Theorem. The closure is R^∞ alongside its limit points.

Section 21

3

(a)

For $\rho(x, x)$, we have $\forall i\ d_i(x, x) = 0$, hence their maximum is 0.

For $\rho(x, y) = 0$, we have some $d_i(x, y) = 0$, hence $x = y$.

We know $\forall i\ d_i(x, y) \geq 0$, so their maximum is at least 0, hence $\rho(x, y) \geq 0$.

We know $\forall i\ d_i(x, y) = d_i(y, x)$, so $\rho(x, y) = \max_i\{d_i(x, y)\} = \max_i\{d_i(y, x)\} = \rho(y, x)$.

Observe $\rho(x, y) = \max\{d_i(x, y)\} \leq \max\{d_i(x, z) + d_i(z, y)\} \leq \max\{d_i(x, z)\} + \max\{d_i(z, y)\} = \rho(x, z) + \rho(z, y)$.

(b)

For $D(x, x)$, we have $d_i(x, x) = 0$, so $\overline{d_i}(x, x)/i = 0$, and their supremum is 0.

If $D(x, y) = 0 = \sup_i\{\overline{d_i}(x, y)/i\}$, then $d_i(x, y) = 0$, since $\overline{d_i}(x, y)/i \geq 0$. Hence $x = y$.

We know some $d_i(x, y) \geq 0$, so $\overline{d_i}(x, y)/i \geq 0$, hence the supremum is at least 0.

Since $d_i(x,y) = d_i(y,x)$ so does $\overline{d_i}(x,y) = \overline{d_i}(y,x)$, and in turn their supremum. i.e $D(x,y) = D(y,x)$.

Observe $D(x,y) = \sup\{\overline{d_i}(x,y)/i\} \leq \sup\left\{\frac{\overline{d_i}(x,z)}{i} + \frac{\overline{d_i}(z,y)}{i}\right\} \leq \sup\left\{\frac{\overline{d_i}(x,z)}{i}\right\} + \sup\left\{\frac{\overline{d_i}(z,y)}{i}\right\} = D(x,z) + D(z,y)$.

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Follows trivially by the author's hints alongside *theorem 21.3*. For example, $x_n + y_n = f(x_n \times y_n) \to f(x \times y) = x + y$.