Homework 05

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Section 30

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Denote the countable basis by $\mathcal{B}$.

For a non-limit point $x \in A$, by definition there is an open $U \ni x : U \cap (A - \{x\}) = \phi$. Moreover, there is a basis element $B : x \in B \subset U$.

For a non-limit point $x' \neq x$, similarly we get a $B' : x' \in B' \subset U'$. It follows $B \neq B'$. So distinct non-limit points of A do induce distinct basis elements of the countable $\mathcal{B}$. Thereby, countably-many points in A are non-limit points of A, implying uncountably-many points in A, are limit points of A.

5a

Call the dense subset S. Construct $\mathcal{B} = \bigcup_{x \in S}\{B(x, 1/n) \mid n \in \mathbb{N}^*\}$. It is countable since the countable union of countable sets is countable. We claim $\mathcal{B}$ is a basis.

Take arbitrary $x \in X$ with $B(x, \epsilon)$. We know there is n_0 where $\frac{1}{n_0} \leq \epsilon$. By density there is $s_0 \in S$ where $d(x, s_0) < \frac{1}{4n_0}$. Note $B(s_0, \frac{1}{2n_0})$ contains x. It is also contained in $B(x, \epsilon)$ since by triangular inequality, any element in it is at distance from x, at most $\frac{1}{4n_0} + \frac{1}{2n_0} = \frac{3}{4n_0} < \frac{1}{n_0} \leq \epsilon$.

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Second-countable

Assume X is second-countable. Let the countable basis of X to be $\mathcal{B}$. Construct countable $\mathcal{B}' = \{f(B) \mid B \in \mathcal{B}\}$. We claim $\mathcal{B}'$ is a basis.

Consider any open $U' \ni f(x)$ in $f(X)$. By hypothesis, $f^{-1}(U') \ni x$ is open in X. Then there is a basis element B whereby $x \in B \subset f^{-1}(U')$. It follows $f(x) \in f(B) \subset U'$ where $f(B)$ is open by hypothesis.

First-countable

If X is first-countable, then for any $f(x) \in f(X)$, we know $x \in X$ by hypothesis has a countable collection $\mathcal{B}$ where any open neighbourhood U of x contains some $B \in \mathcal{B}$. The set $\{f(B) \mid B \in \mathcal{B}\}$ is a countable basis at $f(x)$. The proof is similar.

Section 31

1

Take points $x \neq y$. Since a regular space is also Hausdorff, there are open $U \ni x$ and $U' \ni y$ such that $U \cap U' = \phi$. By *lemma 31.1*, there are open $V \ni x$ and $V' \ni y$ such that $\overline{V} \subset U$ and $\overline{V}' \subset U'$. It follows $\overline{V} \cap \overline{V}' = \phi$. ∎

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If $\mathcal{T}$ is Hausdorff, then trivially so is $\mathcal{T}'$.

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It suffices to prove all limit points of $\{x \mid f(x) = g(x)\}$ are contained in it. Assume towards contradiction there is a limiting point x' such that $f(x') \neq g(x')$. By hypothesis there are disjoint open $V \ni f(x')$ and $V' \ni g(x')$. Consider open neighbourhood $U = f^{-1}(V) \cap g^{-1}(V')$ of x' in X and observe any $z \in U$ satisifes $f(z) \neq g(z)$ as otherwise $f(z) \in V \cap V'$. That U violates x' being a limit point. Contradiction. ∎