Homework 08

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Section 26

1

(a). If X is compact under $\mathcal{T}'$ then it is compact under $\mathcal{T}$.

Take open covering of $\mathcal{T}$. Then by hypothesis it is in $\mathcal{T}'$, and hence there is a finite covering subcollection.

(b). We show $\mathcal{T}' \supset \mathcal{T}$ implies $\mathcal{T} \subset \mathcal{T}'$.

Take arbitrary $U \in \mathcal{T}'$. Then $X - U$ is closed under $\mathcal{T}'$. By *theorem 26.2*, $X - U$ is compact under $\mathcal{T}'$. By *(a)*, $X - U$ is compact under $\mathcal{T}$. By *theorem 26.3*, $X - U$ is closed under $\mathcal{T}$, concluding U is open under $\mathcal{T}$.

2 (b)

Not compact.

Consider open sets $U_k = \mathbb{R} - \{1/i \mid i \geq k\}$. Clearly for any $1/i$, there is a k such that $U_k \ni 1/i$. Hence $\bigcup_k U_k$ covers $[0, 1]$. However, if we took any finite sub-collection, then by considering the maximum index k of them, we know $1/k$ is in $[0, 1]$ but not in the unions of that subcollection. ■

5

By *Lemma 26.4*, for each $y \in B$, we can take disjoint $U_y \ni y$ and $V_y \supset A$. Then $\bigcup_y U_y$ covers B and by its compactness, we take finite sub-collection $U_1, \ldots, U_m$. Set

$$U = U_1 \cup \cdots \cup U_m$$
$$V = V_1 \cap \cdots \cap V_m$$

Observe open $U \supset B$ and open $V \supset A$. Since $U_i \cap V_i = \phi$, it follows $U \cap V = \phi$.

8

($\leftarrow$) Take open neighbourhood V of $f(x_0)$. Note if V has no such point, then $f^{-1}(V) = \phi$ open in X. By definition $Y - V$ is closed. It follows $X \times (Y - V)$ is closed in $X \times Y$ and $G_f \cap X \times (Y - V)$ is closed. Applying *exercise 7*, the set $\{x \mid f(x) \in Y - V\}$ is closed, implying the complement $\{x \mid f(x) \in V\} = f^{-1}(V)$ is open.

($\rightarrow$) Unsolved.

Section 27

3 (a)

Observe the following are open in $[0,1]$ as a subspace.

- $((-1,2)-K)\cap[0,1]=[0,1]-K$.
- $\forall a>0,\ \ (a,2)\cap[0,1]=(a,1]$.

Observe these sets constitute an open cover of $[0,1]$ as a subspace. If we took any finite subcollection, then either we miss 0 or we miss some $1/n$. ■

Section 28

2

Consider the infinite set $A=\{1-1/n \mid n\in\mathbb{N}^*\}$. Consider any $x\in[0,1]-A$.

- Case $x=1$. Then $[1,2)\cap[0,1]=\{1\}$ is open in $[0,1]$ as a subspace, yet does not intersect A.
- Case $0<x<1$. Then $1/n_1<x<1/n_0$ for a smallest n_1 and a biggest n_0. Take $[x,b)$ where $b<1/n_0$, which is open in $[0,1]$ as a subspace, yet does not intersect A.

Thereby, infinite set A does not contain any limit point in $[0,1]$ as a subspace.

3

(a). Yes. Consider an infinite set A in $f(X)$. for each $f(x)\in f(X)$, choose a single point $f^{-1}(x)$ in X, and construct set A^{-1}. Since f is a function, A^{-1} is infinite in X. Observe f restricted to A^{-1} is injective.

By hypothesis, there is a limit point x in A^{-1}. We claim $f(x)$ is a limit point in A. For any neighbourhood $U\ni f(x)$, and since f is continuous, $f^{-1}(U)$ is open in X. Hence some $x'\in A^{-1}\cap f^{-1}(U)$ where $x'\neq x$. It follows $f(x')\in A\cap U$ where $f(x')\neq f(x)$ by injectivity. ■

(b). Yes. Consider an infinite set $B\subseteq A$. By hypothesis, there is a limit point x in B with respect to X as a space. We claim x is a limit point in B with respect to A as a subspace. For open $U\ni x$ in A, we know $U=U'\cap A$ for open U' in X. It follows some $x'\in U'\cap A$ where $x'\neq x$, implying $x'\in U$. ■

4

Partially Solved.

($\leftarrow$) We prove if X is not countably compact, then it is not limit point compact.

By hypothesis, there is a countable cover $\{U_n\}$, which has no finite sub-cover. Take $x_n \notin U_1 \cup \cdots \cup U_n$ for each n. If those $\{x_n\}$ are finite then we can take some x_m such that $x_m \notin \bigcup_n U_n$, concluding $\{x_n\}$ is infinite.

If $\{x_n\}$ has a limit point y, then call open $U_k \ni y$ where $U_k \in \{U_n\}$. Since X is T_1 it follows U_k contains infinitely many points of $\{x_n\}$. But by definition $\forall n \geq k, x_n \notin U_k$, implying U_k contains finitely many points of $\{x_n\}$.