Homework 09

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Section 23

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Assume X is connected with respects to $\mathcal{T}'$. Clearly, if there is a separation in $\mathcal{T}$, so would it exist in $\mathcal{T}'$. Thereby X is connected with respects to $\mathcal{T}$.

On the other hand, X is connected in $\mathcal{T} = \{\phi, X\}$, but for other topologies $\mathcal{T}'$, it may not be connected.

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We use the fact X is connected iff the only open subsets are ϕ and X.

Let U be both open and closed. Then either $X - U$ finite or $U = \phi$ (1). Moreover, $X - U$ is open. Then either $X - (X - U) = U$ is finite or $U = X$ (2).

Assume towards contradiction U is neither ϕ nor X. Then by (1) and (2), we get both $X - U$ and U are finite, implying X is finite. Contradiction.

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If we took subspace $B = \phi$, which is not a one-point set, then it is trivially connected.

We show for any subspace B where $|B| \geq 2$, it is not connected. Take $b_1 \in B$, and set $C = \{b_1\}$ and $D = B - C$. Clearly both are non-empty open sets in B as a subspace, hence constitute a separation.

The converse holds if X is finite. For fixed $x \in X$, we have open U_y containing x but not y for every $y \in X - \{x\}$. Finite intersection $\bigcap_y U_y = \{x\}$ is open. Since all singletons are open, X has the discrete topology, by taking arbitrary unions.

I guess the converse does not hold in general.

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Not connected. We extend example 6 in Munkres but for the uniform topology. Recall in any metric, including the uniform metric, ϵ-balls for $\epsilon < 1$ are open. Take A to be the set of bounded sequences and B the set of unbounded sequences. They are disjoint. For any $x \in A$, we have an open ball $B(x, 1) \subset A$. Similarly for any $x \in B$.

Section 24

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