Homework 1

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Exercises

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(b) → (a). For arbitrary ϵ we are given a subdivision $z_0(\epsilon)$ with $|z_0(\epsilon)| < \delta_0$ for some δ_0 such that $|R(f, z_0(\epsilon) - A)| < \epsilon$. For any finer subdivision w, $|w| \leq |z_0(\epsilon)| < \delta_0$. By hypothesis $|R(f, w) - A| < \epsilon$. ■

(a) → (b).

Insight. The idea is, given a subdivision z, we can take a small enough modulus for z_0, so that it has subdivision points arbitrarily close to points z. With boundedness, the Riemann integral of both shall be close.

Proposition 1. If f is bounded on $[a, b]$ then $\exists M'$ such that $|f(t) - f(y)| \leq M'$ for arbitrary $t, y \in [a, b]$.

Proof. Since f is bounded, the $sup_{t\in[a,b]}\{f(t)\} = M$ exists. Thereby $|f(t) - f(y)| \leq |f(t)| + |-f(y)| = |f(t)| + |f(y)| \leq M + M = M'$.

Definition. For a subdivision z, define $d_{min}(z) = \min\{x_k - x_{k-1}\}$.

Procedure 2. Given any subdivisions $z = (x_0, x_1, \ldots, x_n)$ and w where $|w| \leq d_{min}(z)/2$. Assign to each x_i a nearest y_i in w. It follows $|x_i - y_i| < d_{min}(z)/4$ and in turn no y_i will be assigned twice.

Lemma. $\forall\epsilon\forall z = (x_0, \ldots, x_n)\exists\delta > 0$, s.t if $w = (y_0, \ldots, y_n)$ is a subdivision with $|x_k - y_k| < \delta$ then $|R(f, z \vee w) - R(f, (z - \{x_q\} + \{y_q\}) \vee w)| < \epsilon$ for any $x_q \in x$.

Proof. Observe $(z - \{x_q\} + \{y_1\}) \vee w = (z - \{x_q\}) \vee w$.

Set $\delta = \epsilon/M'$. Let $w = (y_i)$ be any subdivision such that $|x_k - y_k| < \delta$.

Call y_q the point nearest to x_q. If $y_q > x_q$ consider x_{q-1} and if $y_q < x_q$ consider x_{q+1}. WLOG consider the former. Denote $\xi_0 \in [a, x_q]$ and $\xi_1 \in [x_q, y_q]$. When x_q gets removed, these two are replaced by $\xi_0 \in [a, y_q]$. In other words, $\xi_0(x_q - a) + \xi_q(y_q - x_q)$ in $R(f, z \vee w)$ will be $\xi_0(y_q - a)$ in $R(f, z - \{x_q\} \vee w)$.

Therefore,

$$\begin{aligned}
|R(f, z \vee w) - R(f, z \vee w - x_q)| &= |\xi_0(x_q - a) + \xi_q(y_q - x_q) - \xi_0(y_q - a)| \\
&= |\xi_0(x - a - y_q + a) + \xi_1(y_q - x_q)| \\
&= |\xi_1(y_q - x_q) - \xi_0(y_q - x_q)| \\
&= |(\xi_1 - \xi_0)(y_q - x_q)| < |(\xi_1 - \xi_0) \cdot \epsilon/M'| \leq |\epsilon| = \epsilon
\end{aligned}$$

Lemma. $\forall\epsilon > 0\forall z = (x_0, x_1, \ldots, x_n)\exists\delta$, s.t if $w = (y_0, y_1, \ldots, y_n)$ with $|x_k - y_k| < \delta$ then $|R(f, z \vee w) - R(f, w)| < \epsilon$.

Proof. Consider an arbitrary $\epsilon > 0$. Substitute ϵ/n in the previous lemma. Apply the lemma n times substituting x_q by y_q in z to get $\forall k = 1, \dots, n$

$$|R(f, z - \{x_{\pi_1}, \dots, x_{\pi_{k-1}}\} \vee w) - R(f, z - \{x_{\pi_1}, \dots, x_{\pi_k}\} \vee w)| < \epsilon/n$$

By the triangular inequality $|R(f, z \vee w) - R(f, w)| \leq \epsilon/n + \epsilon/n + \cdots + \epsilon/n = \epsilon$.

Corollary 3. $\forall \epsilon > 0 \forall z = (x_0, \dots, x_n) \exists \delta > 0$, s.t if w' is any subdivision finer than $w = (y_0, \dots, y_n)$ with $|x_k - y_k| < \delta$, then $|R(f, z \vee w') - R(f, w')| < \epsilon$.

Theorem. Main problem.

Let f be Riemann integrable. By *proposition (1)* we take M'.

Take arbitrary $\epsilon > 0$. By hypothesis we are given a subdivision $z = (x_0, \dots, x_n)$, s.t $\forall w$ finer, $|R(f, w) - A| < \epsilon/2$.

In *corollary (3)* substitute $\epsilon/2$ and z to get δ_0. Take $\delta = \min\{\delta_0, d_{min}(z)/2\}$. Consider an arbitrary subdivision z_0 with $|z_0| < \delta$. By *procedure (2)* we are given a coarser subdivision $w = (y_0, y_1, \dots, y_n)$ in z_0, s.t $|x_k - y_k| < \delta/2 < \delta$. By *corollary (3)*, $|R(f, z \vee z_0) - R(f, z_0)| < \epsilon/2$.

By triangular inequality

$$\begin{aligned} |R(f, z_0) - A + R(f, z \vee z_0) - R(f, z \vee z_0)| &\leq |R(f, z_0) - R(f, z \vee z_0)| + |R(f, z \vee z_0) - A| \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \blacksquare \end{aligned}$$

Limit equivalence. No. A counter-example is the Dirichlet function. Any value of the form $a + \dfrac{k(b-a)}{n}$ is rational.

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(i)

Following the given hint

$$\begin{aligned} 2 \sin x \sin(\frac{b-a}{n}) &= \cos(x - \frac{b-a}{n}) - \cos(x + \frac{b-a}{n}) \\ 2 \sin(a + k \cdot \frac{b-a}{n}) \sin(\frac{b-a}{n}) &= \cos(a + (k-1)\frac{b-a}{n}) - \cos(a + (k+1) \cdot \frac{b-a}{n}) \end{aligned}$$

Denote $g(k) = \cos(a + k \cdot \frac{b-a}{n})$. Then

$$\begin{aligned}
&\sum_{k=1}^{n} \cos(a + (k-1)\frac{b-a}{n} - \cos(a + (k+1)\frac{b-a}{n} \\
&= \sum_{k=1}^{n} g(k-1) - g(k+1) \\
&= g(0) - g(2) \\
&+ g(1) - g(3) \\
&+ .. \\
&+ g(n-2) - g(n) \\
&+ g(n-1) - g(n+1) \\
&= g(0) + g(1) - g(n) - g(n+1)
\end{aligned}$$

It follows

$$\begin{aligned}
&\sum_{k=1}^{n} 2\sin(a + k \cdot \frac{b-a}{n}) \sin(\frac{b-a}{n}) \\
&= 2\sin(\frac{b-a}{n}) \sum_{k=1}^{n} \sin(a + k \cdot \frac{b-a}{n}) \\
&= \cos(a) + \cos(a + \frac{b-a}{n}) - \cos(b) - \cos(a + \frac{n+1}{n} \cdot (b-a))
\end{aligned}$$

Observe

$$\lim_{n\to\infty} \cos(a + \frac{b-a}{n}) = \cos(a)$$
$$\lim_{n\to\infty} \cos(a + \frac{n+1}{n} \cdot (b-a)) = \cos(a + b - a) = \cos(b)$$

Thereby

$$\sum_{k=1}^{n} \sin(a + k \cdot \frac{b-a}{n}) = 1/\sin(\frac{b-a}{n}) \cdot (\cos(a) - \cos(b))$$

(ii) Not solved

The Riemann sum is

$$\sum_{k=1}^{n} \frac{b-a}{n} \sin(a+k\cdot\frac{b-a}{n}) = \frac{b-a}{n} \sum_{k=1}^{n} \sin(a+k\cdot\frac{b-a}{n}) = \frac{n}{(b-a)\sin(\frac{b-a}{n})}(\cos(a)-\cos(b))$$

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(i) The statement does not hold in general. As a counter-example, consider $f : [0, 1] \to \mathcal{R}$ where $f(x) = 1/x$ and $x \neq 0$. For any $[0, x_1]$, we can choose ξ_1, so that $|f(x) \cdot (x_1 - 0) - f(\xi_1) \cdot (x_1 - 0)|$ is lower-bounded by some constant.

(ii) Depending on choices of ξ, each partial sum can be either 0 or 1. Thereby the function $g_{z,t}(x)$ could be 0 or $1 * (b - a) = b - a$ for example.