MACT 4127 - Real Analysis II

Spring 2025

Homework (1)

Deadline for submission: Feb. 20 on CANVAS.

(1) Let $f : [a, b] \longrightarrow \mathbb{R}$. Prove that the following are equivalent:

(a) f is Riemann integrable on $[a, b]$

(b) There exists a real number A satisfying

$$\forall \epsilon > 0\, \exists \delta > 0\, (\forall \mathbf{z} \text{ a subdivision of } [a,b] : |\mathbf{z}| < \delta \Longrightarrow |R(f, \mathbf{z}) - A| < \epsilon)\,.$$

Are these conditions equivalent to the existence of the limit

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(a + \frac{k(b-a)}{n})\,?$$

(2) (i) Find an expression for the Riemann sum of $f(x) = \sin x$ w.r.t. the uniform subdivision of $[a, b]$ which does not include a summation. (**Hint:** Multiply by $\sin(\frac{b-a}{n})$ and use the trigonometric law $\sin u \sin v = \frac{1}{2}(\cos(u-v) - \cos(u+v))$).

(ii) Deduce that $\int_a^b \sin x\, dx = \cos a - \cos b$.

(3) For any function $f : [a, b] \longrightarrow \mathbb{R}$ and any subdivision $\mathbf{z} = (x_0, \cdots, x_n)$ of $[a, b]$ and any choice of points $\mathbf{t} = (t_1, \cdots, t_n)$ with $t_k \in [x_{k-1}, x_k]$, we define a step function

$$g_{\mathbf{z},\mathbf{t}}(x) = \sum_{k=1}^{n-1} f(t_k)\chi_{[x_{k-1},x_k[}(x) + f(t_n)\chi_{[x_{n-1},x_n]}(x), \quad x \in [a, b].$$

(i) Prove that if f is continuous, then

$$\forall \epsilon > 0 \exists \delta > 0 \quad \text{such that for any subdivision } \mathbf{z}: \; |\mathbf{z}| < \delta \Longrightarrow d_\infty(g_{\mathbf{z},\mathbf{t}}, f) < \epsilon, \; \text{for any selection of points } \mathbf{t} = (t_1, \cdots, t_n).$$

(ii) What do you think will the step functions $g_{\mathbf{z},\mathbf{t}}$ be like when f is the Dirichlet function?