Homework 02

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Contents

Exercises

1

Let D be the points of discontinuity. For $\varepsilon > 0$ set $\eta = \min(\varepsilon, \varepsilon/2(b-a))$. Observe D_η is negligible as D is negligible. Take M, m to be an upper and lower bound of f respectively.

By negligibility D_η is coverable by a family $(J_m)_{m\in N^*}$ of open intervals such that $\sum_{m=1}^{\infty} \ell(J_m) < \varepsilon/2(M-m)$. Since it is closed and bounded, by *Hein-Borel* we take a finite cover $J_1, \dots, J_N$ where $J_m =]a_m, b_m[$ and $b_{m-1} \leq a_m$. Observe $D_\eta \subseteq \bigcup_{m=1}^{N} J_m$.

It follows if $x \notin \bigcup_{m=1}^{N} J_m$ then $w(f;x) < \eta$. Then there exists $\delta_x > 0$ such that $w_\delta(f;x) < \eta$. Call the corresponding open set S_x. Then $|\sup S_x - \inf S_x| < \eta$. Observe $(\bigcup_{m=1}^{N} J_m)^{\complement} \subseteq \bigcup_x S_x$. By *Hein-Borel* we can take finite $\{S_x\}$. Among their corresponding $\{\delta_x\}$, take the smallest and call it δ.

Subdivide $[b_{m-1}, a_m]$ into finite intervals $I_1^{(m)}, \dots, I_{r_m}^{(m)}$, each of length less than δ. Take z_ε to be a subdivision of $[a,b]$ containing all I_k^m for $m = 1, \dots, N$ and $k = 1, \dots, r_m$.

Now for any $S_k = [y_{k-1}, y_k]$, either

- **Case 1.** $S_k \subseteq J_m$. Then $M_k - m_k \leq M - m$. Denote those intervals by B.
- **Case 2.** $S_k \subseteq I_r^{(m)}$. Then $M_k - m_k \leq \eta$. Denote those intervals by G.

We show $U(f;w) - L(f;w) < \varepsilon$

$$
\begin{aligned}
U(f;w) - L(f;w) &= \sum_{k=1}^{n} (M_k - m_k)(y_k - y_{k-1}) \\
&= \sum_{k\in G} (M_k - m_k)(y_k - y_{k-1}) + \sum_{k\in B} (M_k - m_k)(y_k - y_{k-1}) \\
&\leq \eta \sum_{k\in G} (y_k - y_{k-1}) + (M-m) \sum_{k\in B} (y_k - y_{k-1}) \\
&\leq \eta(b-a) + (M-m)\frac{\varepsilon}{2(M-m)} \\
&\leq \frac{\varepsilon}{2(b-a)}(b-a) + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

2

Lemma. $\underline{\int}_a^b f(x)dx \leq c_i(S(f))$

For any $L(f;z)$, the corresponding rectangles could be re-interpreted as a finite union of disjoint rectangles, reserving exactly the same area. Thereby

$$\{L(f;z) \mid \text{subdivision } z\} \subseteq \{area(B) \mid B \subseteq A, B \text{ is a simple set}\}$$
$$\sup\{L(f;z) \mid \text{subdivision } z\} \leq \sup\{area(B) \mid B \subseteq A, B \text{ is a simple set}\}$$
$$\int_{\underline{\quad} a}^{b} f(x)dx \leq c_i(S(f))$$

Lemma. $c_i(S(f)) \leq \int_{\underline{\quad} a}^{b} f(x)dx$

Since f is bounded on a closed interval $[a,b]$, $area(S(f)) = \int_{\underline{\quad} a}^{b} f(x)dx$. But for any simple set $B \subseteq S(f)$, we know $area(B) \leq area(S(f))$. It follows

$$\sup\{area(B) \mid \text{simple set B}\} \leq \int_{\underline{\quad} a}^{b} f(x)dx$$
$$c_i(S(f)) \leq$$

3

(i)

Consider a function f induced by a line segment. Then it is bounded on a closed interval, and hence Riemann integrable. Observe $U(f;z) - L(f;z)$ induces rectangles which do cover the line. Since $|U(f;z) - L(f;z)| \to 0$, it follows there are simple sets of disjoint rectangles $S_1, S_2, \ldots$ such that $area(S_i) \to 0$. Therefore, the outer Jordan content is $c_o(PQ) = 0$. Since $c_i(PQ) \leq c_o(PQ)$ it follows also $c_i(PQ) = 0$.

(ii)

Observe $area(R_n) = \left(\frac{1}{2^n} - \frac{1}{2^{n+1}}\right) \cdot n2^{n+1} = n$. Since $\{R_n\}$ is a pairwise disjoint set of rectangles,

$$area(R_1 \cup R_2 \cup \cdots \cup R_n) = area(R_1) + \cdots + area(R_n)$$
$$area\left(\bigcup_{i=1}^{N} R_n\right) = 1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$
$$c_i\left(\bigcup_{i=1}^{N} R_n\right) = c_o\left(\bigcup_{i=1}^{N} R_n\right) =$$

Therefore $c_i\left(U_{i=1}^{\infty} R_n\right) = c_o\left(U_{i=1}^{\infty} R_n\right) = \infty$.

(iii)

3

(i)

Holds. For any A, and p_n, we have $t_n \not< 0$. Hence $\sum_{p_n \in A} t_n \not< 0$.

(ii)

Holds. if $A \subset B$, then

$$\begin{aligned}
\sum_{p_n \in B} t_n &= \sum_{p_n \in A} t_n + \sum_{p_n \in B-A} t_n && \text{since } A \cap (B - A) = \phi \\
&= v(A) + v(B - A) \\
&\geq v(A) && \text{since } v(B - A) \geq 0 \text{ by } (i)
\end{aligned}$$

(iii)

Holds.

$$\begin{aligned}
v(A \cup B) &= \sum_{p_n \in A \cup B} t_n \\
&= \sum_{p_n \in A} t_n + \sum_{p_n \in B} t_n && \text{since } A \cap B = \phi \\
&= v(A) + (B)
\end{aligned}$$

(iv)