MACT 4127 - Real Analysis II

Spring 2025

Homework (2)

Deadline for submission: March 13 on CANVAS.

(1) Prove (directly, without using the Lebesgue-Vitali theorem) that any bounded function on $[a, b]$ with a countable number of points of discontinuity is Riemann integrable.

(2) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a non-negative bounded function, and let

$$S(f) = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], 0 \le y \le f(x)\}.$$

Prove that

$$c_i(S(f)) = \underline{\int_a^b} f(x)\, dx, \quad \text{and } c_o(S(f)) = \overline{\int_a^b} f(x)\, dx.$$

Deduce that f is Riemann integrable iff $S(f)$ is rectifiable.

(3) Find the inner and outer Jordan content for each of the following sets:

(i) A line segment PQ.

(ii) The set

$$B = \cup_{n=1}^{\infty} R_n, \text{ where } R_n = \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right) \times \left[0, n2^{n+1}\right] : n \in \mathbb{N}^*.$$

(iii) The disc centered at the origin and with radius 1.

(4) Let $\{p_n : n \in \mathbb{N}^*\}$ be an enumeration of the points with rational coordinates in $\mathbb{R}^2$, and let $(t_k)_{k\in\mathbb{N}^*}$ be a sequences of strictly positive numbers. Define the set function:

$$\nu : \mathcal{P}(\mathbb{R}^2) \longrightarrow \mathbb{R} \cup \{+\infty\}, \quad \text{by } \nu(A) = \sum_{p_n \in A} t_n.$$

Examine ν for the following properties:

(i) $\nu(A) \ge 0$ for all A.

(ii) ν is monotone, *i.e.*

$$A \subset B \Longrightarrow \nu(A) \le \nu(B).$$

(iii) ν is additive, *i.e.*

$$\nu(A \cup B) = \nu(A) + \nu(B), \quad \forall A, B \subset \mathbb{R}^2, A \cap B = \emptyset.$$

(iv) ν is σ-additive, *i.e.*

$$\nu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n),$$

for any family $(A_n)_{n \in \mathbb{N}^*}$ of pairewise disjoint subsets of $\mathbb{R}^2$