Homework 03

Mostafa Touny

April 15, 2025

Contents

Exercises

1

Lemma. $\mu(\limsup_n A_n) \geq \limsup_n \mu(A_n)$

Set $A'_n = \bigcup_{k\geq n} A_k$ and observe it is a decreasing sequence $A'_n \supseteq A'_{n+1}$ where $\mu(A'_1) < \infty$ by hypothesis. It follows

$$\mu(\bigcap_{n=1}^{\infty} A'_n) = \lim_{n\to\infty} \mu(A'_n) = \lim_{n\to\infty} \mu(\bigcup_{k\geq n} A_k) \ (1)$$

$$\mu(\bigcap_{n=1}^{\infty}(\bigcup_{k\geq n} A_k)) =$$

Fix n and observe $\forall k \geq n$ $\mu(\bigcup_{k\geq n} A_k) \geq \mu(A_k)$ as $\forall k \geq n$ $\bigcup_{k\geq n} A_k \supseteq A_k$. It follows for any n, $\mu(\bigcup_{k\geq n} A_k) \geq \sup_{k\geq n} \mu(A_k)$. Therefore $\lim_{n\to\infty} \mu(\bigcup_{k\geq n} A_k) \geq \lim_{n\to\infty} \sup_{k\geq n} \mu(A_k)$ (2).

By (1) and (2), the intended result follows.

Lemma. $\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n)$

Set $A'_n = \bigcap_{k\geq n} A_k$ and observe it's increasing $A'_k \subseteq A'_{k+1}$. Then

$$\mu(\bigcup_{n=1}^{\infty} A'_n) = \lim_{n\to\infty} \mu(A'_n)$$

$$\mu(\bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k) = \lim_{n\to\infty} \mu(\bigcup_{k\geq n} A_k) \ (1)$$

Clearly $\forall n \forall k$, $\bigcap_{k\geq n} A_k \subseteq A_k$, implying $\mu(\bigcap_{k\geq n} A_k) \leq \mu(A_k)$. It follows $\forall n$ $\mu(\bigcap_{k\geq n} A_k) \leq \inf_{k\geq n} \mu(A_k)$. Hence

$$\lim_{n\to\infty}(\mu(\bigcap_{k\geq n} A_k)) \leq \liminf_k \mu(A_k) = \lim_{n\to\infty} \inf_{k\geq n} A_k \ (2)$$

By (1) and (2), the intended result follows.

Example. Equality.

Take $(A_k)_k$ pairwise disjoint where $\mu(A_k) \to 0$. Then $\bigcap_{n=1}^{\infty} \cup_{k\geq n} A_k = \phi$ where $\mu(\phi) = 0$, for the left hand side. On the right hand side, $\limsup_n \mu(A_n) = 0$.

Lemma. Borel-Cantelli.

Since $\sum_{n=1}^{\infty}\mu(A_n) = c$ for some constant c, clearly

$$\begin{aligned}
\forall n \ \sum_{k\geq n}^{\infty}\mu(A_k) &= c - \sum_{k=1}^{n-1}\mu(A_k) \\
\lim_{n\to\infty}\sum_{k\geq n}^{\infty}\mu(A_k) &= c - \lim_{n\to\infty}\sum_{k=1}^{n-1}\mu(A_k) \\
&= c - c \\
&= 0 \quad (1)
\end{aligned}$$

By measure's sub-additivity

$$\begin{aligned}
\forall n \ \mu(\bigcup_{k\geq n} A_k) &\leq \sum_{k\geq n}^{\infty}\mu(A_k) \\
\lim_{n\to\infty}\mu(\bigcup_{k\geq n} A_k) &\leq \lim_{n\to\infty}\sum_{k\geq n}^{\infty}\mu(A_k) \quad (2)
\end{aligned}$$

By (1) and (2), $\mu(\bigcap_{n=1}^{\infty}\cup_{k\geq n}A_k) = \lim_{n\to\infty}\mu(\bigcup_{k\geq n} A_k) \leq 0$. ■

2

ϕ is not in the domain, so the set of singletons is not a semi-ring.

3

Partially Solved

Note we are given $\mu(\Omega) = v(\Omega)$.

Lemma. $\mu(E^c) = v(E^c)$.

$$\begin{aligned}
\mu(E) &= v(E) \\
\mu(\Omega \setminus E^c) &= v(\Omega \setminus E^c) \\
\mu(\Omega) - \mu(E^c) &= v(\Omega) - v(E^c) \\
\mu(E^c) &= v(E^c)
\end{aligned}$$

4

Partially Solved

a.

Lemma. $\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$

It suffices to show, the generating set of the L.H.S is a subset of the generating set of the R.H.S. Let S be an arbitrary open set of $\mathbb{R}^2$. Then we can take points $p_i \in S \cap \mathbb{Q}$ covered by open balls. Take rectangles R_i subset of those open balls, and note $p_i \in R_i \subseteq S$ implying the countable union $\bigcup_i R_i = S$ by the density of rationals. Denote $R_i = [a_i, b_i[\times [c_i, d_i[$. Since $[a_i, b_i[, [c_i, d_i[\in \mathcal{B}(\mathbb{R})$ it is clear R_i is in the generating set of $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ implying $\bigcup_i R_i \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

Lemma. For any open sets A and B in $\mathbb{R}$, it follows $A \times B \in \mathcal{B}(\mathbb{R}^2)$.

Take any element $(a, b) \in A \times B$ and observe there are open neighbourhoods whereby $a \in]x, x'[\subseteq A$ and $b \in]y, y'[\subseteq B$. Then $R_{a,b} =]x, x'[\times]y, y'[\subseteq A \times B$ and $\bigcup_{a,b} R_{a,b} = A \times B$. Since each $R_{a,b}$ is open in $\mathbb{R}^2$, it follows $A \times B \in \mathcal{B}(\mathbb{R}^2)$.

b.

By a, measure $(m \otimes m)$ is defined on $\mathcal{B}(\mathbb{R}^2)$. For any element $[a, b[\times [c, d[$ of the generating set of $\mathcal{B}(\mathbb{R}^2)$, we know $m^2([a, b[\times [c, d[) = (b-a)(d-c) = m([a, b[) \cdot m([c, d[) = (m \otimes m)([a, b[\times [c, d[)$. By Caratheodory's uniqueness, the intended result follows.

5

Unsolved