MACT 4127 - Real Analysis II

Spring 2025

Homework (3)

Deadline for submission: April 15 on CANVAS.

---

(1) Let (Ω, Σ, μ) be a finite measure space, and let $(A_k)_{k\in\mathbb{N}^*}$ be a family of sets in Σ. We define

- $\limsup_n A_n = \cap_{n=1}^{\infty} (\cup_{k\geq n} A_k)$,
- $\liminf_n A_n = \cup_{n=1}^{\infty} (\cap_{k\geq n} A_k)$.

Prove that

$$\mu(\limsup_n A_n) \geq \limsup_n \mu(A_n), \quad \text{and } \mu(\liminf_n A_n) \leq \liminf_n \mu(A_n).$$

Give examples to show that equalities do not hold.

Use the first inequality to prove:

The Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$, then $\mu(\limsup_n A_n) = 0$.

---

(2) Let $\lambda > 0$ be fixed, and let $p_n = \dfrac{\lambda^n}{n!}$. Define the set function:

$$p_{\lambda,0}(\{k\}) = p_k, \quad k \in \mathbb{N}^*.$$

(i) Show that $p_{\lambda,0}$ is a premeasure.

(ii) Apply the Carathéodory extension theorem to obtain a measure p_λ that extends $p_{\lambda,0}$.

(iii) What is the domain of p_λ?

(iv) Is p_λ a finite measure?

---

(3) Let (Ω, Σ) be a measurable space, and let $\mathcal{E} \subset \Sigma$ be a generating set,*i.e.* $\Sigma = \sigma(\mathcal{E})$, such that $\Omega \in \mathcal{E}$.
Prove that if μ, ν are two finite measures defined on Σ satisfy $\mu(E) = \nu(E)$ for any $E \in \mathcal{E}$, then $\mu(A) = \nu(A)$ for all $A \in \Sigma$.

---

(4) Prove that the Lebesgue measure on $\mathbb{R}^2$ is the product of two copies of the Lebesgue measure on $\mathbb{R}$, all considered on Borel sets, *i.e.* prove that

(a) $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$,

(b) For any $B \in \mathcal{B}(\mathbb{R}^2)$: $m_2(B) = (m \otimes m)(B)$.

---

(5) Prove that if μ is a measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, which is invariant under translations, (*i.e.* $\mu(B + (x_0, y_0)) = \mu(B)$ for any Borel set B and any $(x_0, y_0) \in \mathbb{R}^2$), and $\mu(B) < +\infty$ for any bounded Borel set B, then there exists a positive constant c such that $\mu(B) = cm(B)$ for any Borel set B.

Deduce that the Lebesgue measure on $\mathbb{R}^2$ is invariant under rotations.

The next part will not be graded, it is added just for fun :)

(6) **(Hausdorff Measures)** We fix a real number $\alpha \geq 0$. The object of this exercise is to define a measure on Borel subsets of $\mathbb{R}^N$ called the α-Hausdorff measure.
A ***canonical*** r***-box*** in $\mathbb{R}^N$ is a set of the form

$$B(a; r) = [a_1, a_1 + r[\times [a_2, a_2 + r[\times \cdots \times [a_N, a_N + r[, \quad \text{where } a = (a_1, a_2, \cdots, a_N) \in \mathbb{R}^N.$$

Given $\delta > 0$ we define the set function

$$H^\alpha_\delta(A) = \inf\{\sum_{k=1}^{\infty} r_k^\alpha : A \subset \cup_{k=1}^{\infty} B(a^{(k)}; r_k), 0 \leq r_k < \delta \, \forall k\}.$$

(i) Prove that if $\delta_1 < \delta_2$, then $H^\alpha_{\delta_1}(A) \geq H^\alpha_{\delta_2}(A)$.

(ii) Define $H^{\alpha*}(A) = \lim_{\delta \to 0} H^\alpha_\delta(A)$, and prove that it's an outer measure.

(iii) For $N = 1$: compute $H^{\alpha*}(A)$ for $\alpha = 0, \frac{1}{2}, 1$ where A is the unit interval. Do the same when A is a finite set of points and when $A = \mathbb{Q}$.

(iv) We say that $A \subset \mathbb{R}^N$ is H^α-measurable if

$$H^{\alpha*}(E) = H^{\alpha*}(E \cap A) + H^{\alpha*}(E \setminus A), \text{ for all } E \subset \mathbb{R}^N.$$

Prove that the family $\mathcal{M}^\alpha$ of all H^α-measurable sets is a σ-algebra, that contains all Borel sets.

(v) H^α is defined to be the restriction of $H^{\alpha*}$ to $\mathcal{M}^\alpha$.
Prove that H^α is a measure. (You only need to show that it's σ-additive).

(vi) Given a Borel set A, show that there exists $\alpha_0 \geq 0$ such that

$$\alpha < \alpha_0 \Longrightarrow H^\alpha(A) = +\infty, \quad \alpha > \alpha_0 \Longrightarrow H^\alpha(A) = 0.$$

This value α_0 is called the Hausdorff dimension of A.

(vii) Show that for a non-empty open set $U \subset \mathbb{R}^N$: the Hausdorff dimension of U is N.

(viii) Find (on the internet) an example of a subset of $\mathbb{R}^2$ whose Hausdorff dimension is $\frac{3}{2}$.