MACT 4127 - Real Analysis II

Spring 2025

To be submitted by May 24, 2025

Electronic submission is accepted only in PDF format.

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(1) Let (Ω, Σ, μ) be a measure space, and let $f, g : \Omega \longrightarrow \overline{\mathbb{R}}$.
Prove that if $f.g$ are integrable functions, then

$$f \wedge g = \min(f, g), \text{ and } f \vee g = \max(f, g)$$

are integrable functions.

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(2) Find an example for two integrable functions whose product is not integrable.

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(3) Let (Ω, Σ, μ) be a measure space, and let $f : \Omega \longrightarrow \overline{\mathbb{R}}$ be a Σ-measurable function.

(a) Let f be non-negative, and for any $t > 0$ let

$$A_t = \{x \in \Omega : f(x) \geq t\}.$$

Prove Markov inequality:

$$\mu(A_t) \leq \frac{1}{t} \int f \chi_A \, d\mu.$$

(b) Let $f : \Omega \longrightarrow \overline{\mathbb{R}}$ be an arbitrary Σ-measurable function. Prove that if f is μ-integrable, then there exists a countable family $(B_n)_{n \in \mathbb{N}^*}$ of elements of Σ, each of which has finite measure, such that

$$f^{-1}\left(\overline{\mathbb{R}} \setminus \{0\}\right) = \cup_{k=1}^{\infty} B_k.$$

(c) Prove that if f is μ-integrable, and $\int_\Omega |f| \, d\mu = 0$, then $f = 0$ μ-almost everywhere.

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(4) Let (Ω, Σ, μ) be a finite measure space, and let $f : \Omega \longrightarrow \overline{\mathbb{R}}$ be a non-negative measurable function. Let

$$A_n = \{x \in \Omega : f(x) \geq n\}.$$

Prove that f is μ-integrable iff the series $\sum_{n=1}^{\infty} \mu(A_n)$ is convergent/

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(5) Let (Ω, Σ, μ) be a measure space, and let $f : \Omega \times [a, b] \longrightarrow \mathbb{R}$ be a function satisfying

(i) For each fixed $t \in [a, b]$: the function

$$f_t : \Omega \longrightarrow \mathbb{R} \text{ defined by } f_t(x) = f(x, t)$$

is μ-integrable.

(ii) For each fixed $x \in \Omega$: the function

$$f^x : [a, b] \longrightarrow \mathbb{R} \text{ defined by } f^x(t) = f(x, t)$$

is differentiable.

(iii) There exists an integrable function $g : \Omega \longrightarrow [0, +\infty[$ such that

$$\left| \frac{f(x,t) - f(x,s)}{t - s} \right| \leq g(x), \quad \forall x \in \Omega, \forall t, s \in [a, b], t \neq s.$$

Prove that, for each $t_0 \in]a, b[$, the function $h(t) = \int_\Omega f_t \, d\mu$ is differentiable at t_0, and that

$$\frac{dh}{dt}|_{t=t_0} = \int_\Omega \frac{d}{dt} f(., t)|_{t=t_0} \, d\mu.$$
