Problem-Set 02

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Problem. 1

We prove each axiom as listed by *Rudin* in page 5.

A1 $(a_0 + a_1) + \sqrt{2}(b_0 + b_1) \in \mathcal{Q}(\sqrt{2})$, As $(a_0 + a_1), (b_0 + b_1) \in \mathcal{Q}$.

A2 Follows immediately by properties of $\mathcal{Q}$.

A3 Follows immediately by properties of $\mathcal{Q}$.

A4 $0_{\mathcal{Q}(2)}$ here is the number $0 + \sqrt{2}\,0 = 0_{\mathcal{R}}$.

A5 For an $x_{\mathcal{Q}(2)}$, $-x_{\mathcal{Q}(2)} = -a + \sqrt{2}(-b)$.

M1 The product is $(a_0a_1 + 2b_0b_1) + \sqrt{2}(a_0b_1 + a_1b_0)$, Where the formed a and b are in $\mathcal{Q}$.

M2 Following properties of $\mathcal{Q}$, The product we formed in *M1* is the same in cases of xy and yx.

M3 Following properties of $\mathcal{Q}$, The product we formed in *M1* is the same in cases of $(xy)z$ and $x(yz)$.

M4 $1_{\mathcal{Q}(\sqrt{2})}$ here is $1_{\mathcal{R}} \neq 0_{\mathcal{R}} = 0_{\mathcal{Q}(2)}$.

M5 If $x_{\mathcal{Q}(\sqrt{2})} \neq 0_{\mathcal{Q}(2)} = 0 + \sqrt{2} \cdot 0$, Then we know either $a \neq 0$ or $b \neq 0$, and hence $x_{\mathcal{Q}(\sqrt{2})} = a + b\sqrt{2} \neq 0$. Define $x^{-1}_{\mathcal{Q}(\sqrt{2})} = \dfrac{1}{a + b\sqrt{2}}$. What is remaining is to show $\dfrac{1}{a + b\sqrt{2}} \in \mathcal{Q}(2)$ by a multiplication by its conjugate. Observe:

$$\begin{aligned}
&\frac{1}{a + b\sqrt{2}} \\
=&\frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} \\
=&\frac{a - b\sqrt{2}}{a^2 + 2b^2} = \left(\frac{a}{a^2 + 2b^2}\right) + \left(\frac{-b}{a^2 + 2b^2}\right)\sqrt{2}
\end{aligned}$$

And clearly $(\frac{a}{a^2 + 2b^2}), (\frac{-b}{a^2 + 2b^2}) \in \mathcal{Q}$.

D Follows by a trivial algebra.

Problem. 2

Let's look at the special case of $z = (x, 0)$. Then for any $r > 0$, there exists a complex number $w = (x/r, 0)$, such that $rw = z$.

From now on we focus on $z = (x, y)$ assuming $y \neq 0$. Before proceeding, we develop a central lemma.

Lemma. 1 For any complex number $w = (a, b)$, $|w| = 1 \leftrightarrow a^2 + b^2 = 1$.
Follows immediately by setting $w \cdot \overline{w} = 1$ and multiplying.

Lemma. 2 Given any x and $y \neq 0$, Finding reals r, a, b such that $r \cdot a = x$, $r \cdot b = y$ satisfies $z = (x, y) = r \cdot (a, b) = rw$
Follows immediately by a trivial algebra.

Theorem. 3 Main Problem
Now we combine *Lemma 1* and *Lemma 2* to satisfy both requirements by forming a combined system of equations, Given any $z = (x, y)$ where $y \neq 0$.

$$\begin{aligned} r \cdot a &= x \\ r \cdot b &= y \\ a^2 + b^2 &= 1 \end{aligned}$$

It can be solved by substitution where:

$$\begin{aligned} a &= \sqrt{1 - b^2} \\ r &= y/b \quad \text{valid as b isn't zero} \\ 3/b \cdot \sqrt{1 - b^2} &= x \end{aligned}$$

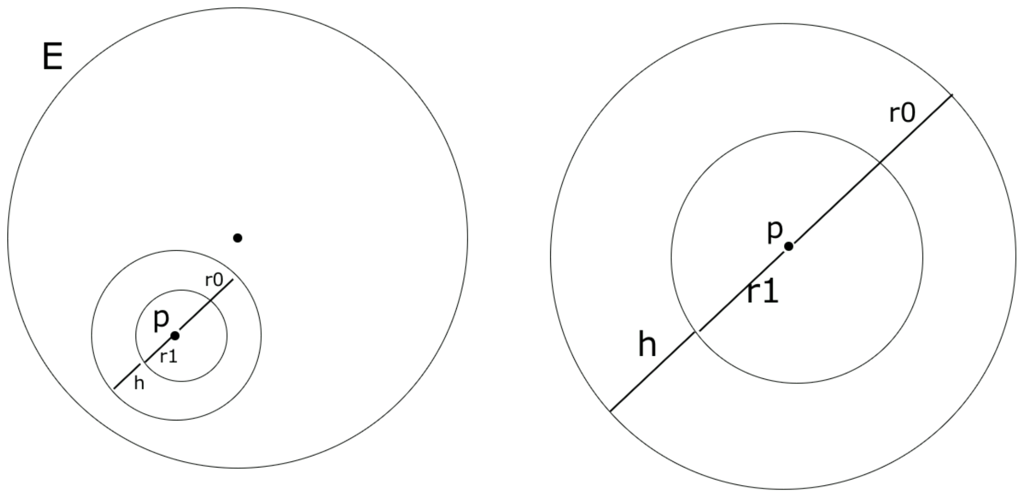
Note $b \neq 0$ lest $r \cdot b = r \cdot 0 = 0 = y$, Contradicting our assumption.

The system uniquely determines the values

$$\begin{aligned} b &= \frac{3}{\sqrt{x^2 + 9}} \\ r &= \frac{y}{3} \cdot \sqrt{x^2 + 9} \\ a &= \sqrt{1 - \frac{9}{x^2 + 9}} \end{aligned}$$

Problem. 3

a



We show if arbitrary $p \in E^o$ then p is an interior of E^o. By definition p is an interior of E. So $N_{r_0}(p) \subset E$ for some $r_0 > 0$. Let $r_1 = r_0/2$ and $h = r_0 - r_1$. It suffices to show $N_{r_1}(p) \subset E^o$.

Consider $N_h(p')$ for any $p' \in N_{r_1}(p)$. Through the picture it is clear this new neighbourhood shall be bounded by $N_{r_0}(p)$ and hence falls completely within E. That shows $p' \in E^o$ and in turn completes our proof.

In greater details, Observe $\forall q \in N_h(p')$, $d(q,p) \leq d(q,p') + d(p',p) < h + r_1 = (r_0 - r_1) + r_1 = r_0$, and hence $q \in N_{r_0}(p) \subset E$.

b

($\leftarrow$) Trivial by a.

($\rightarrow$) Trivially $E^o \subset E$. By hypothesis, The definition of open E immediately concludes $E \subset E^o$.

c

Any $p \in G$ is an interior point of G by definition. So there is a neighbourhood $N_{r_0}(p) \subset G$ for some $r_0 > 0$. But we know $G \subset E$, So $N_{r_0}(p) \subset E$, p is an interior point of E.

e

I guess Yes. We struggled with a formal proof though.

Problem. 4

Definition. 1 Given a point $p \in X$, Define $V_p = \{x > p \mid [p,x] \subset X\} \cup \{x < p \mid [x,p] \subset X\}$.

Remark. 2 V_p constitutes a largest segment (a,b), Given X is an open-set.
Assuming $V_p = (a,b]$ derives an immediate contradiction as b won't be an interior point of X.

A more rigorous argument for showing V_p is a segment can be made by constructing a segment $(inf\, V_p, sup\, V_p)$ but for brevity we ignore it.

Lemma. 3 Given an open-set X and some $V_p \subset X$, For any $q \neq p$, Either $V_p = V_q$ or $V_p \cap V_q = \phi$.
Easily proven by considering the equivalent logical form of $V_p \cap V_q \neq \phi \rightarrow V_p = V_q$.

Lemma. 4 Given a non-empty open-set X and some $V_p \subset X$, $X_1 = X - V_p$ is either empty or a non-empty open-set.
If $V_p = X$ then X_1 is empty. Consider V_p as a strict or proper subset of X. Then X_1 is non-empty.

We show now X_1 is an open-set. Let q be an arbitrary point of X_1, Then also $q \in X$. Since X is an open-set we know there's some neighbour $N_{r0}(q) \subset X$. Clearly $N_{r0}(q) \subset V_q$. By *Lemma 3* and since $q \notin V_p$, It follows $N_{r0}(q) \cap V_p = \phi$. So $N_{r0}(q) \subset X_1$ and q is an interior point of X_1.

Corollary. 5 Countable $\{V_i\}$
Follow the same procedure of *Lemma 4* but let the taken point p_i to be a rational number. Take some real number z_i in non-empty X_i; As it is interior there is a neighbour such that for any q where $d(z_i, q) < r_0$ for some $r_0 > 0$, $q \in X$. By the density of rational numbers, there is a rational p_i which satisfies $d(z_i, p_i) < r_0$. Hence $p_i \in X_i$.

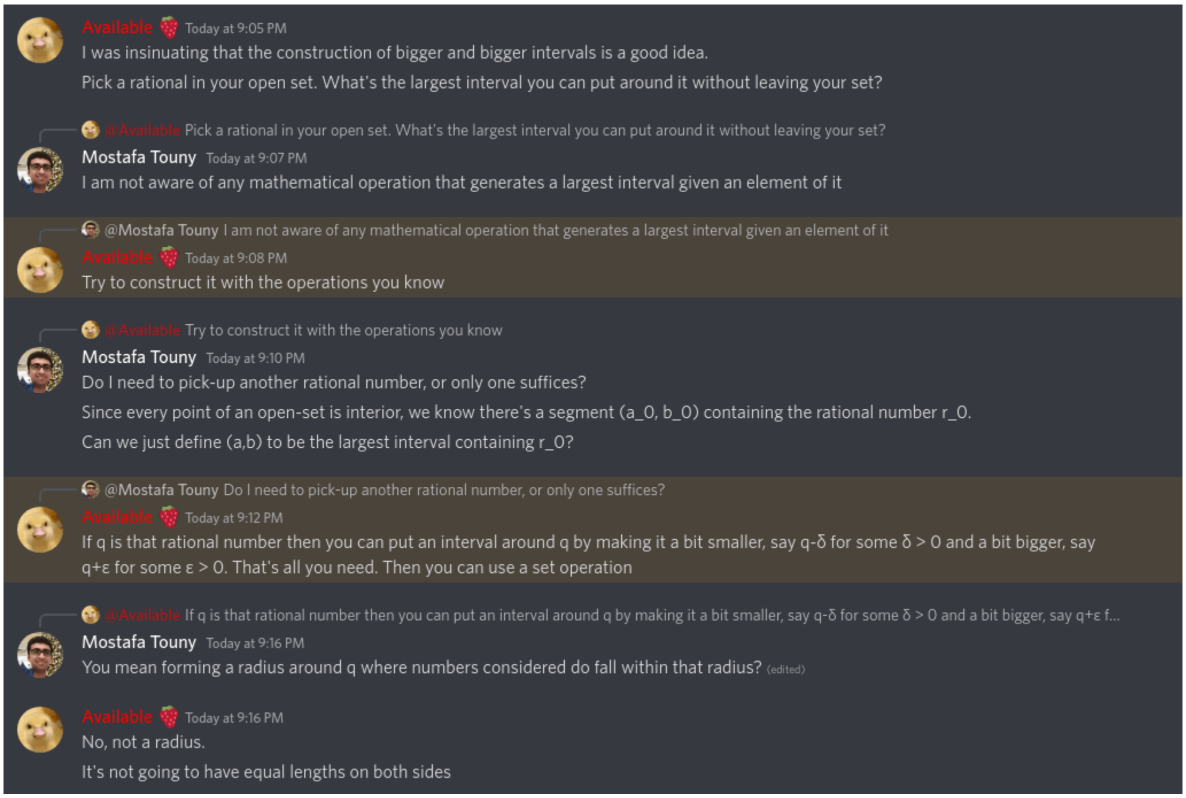
We now know every distinct V_{p_i} corresponds to a distinct rational number p_i. So the cardinality of $\{V_{p_i}\}$ is *at most countable.*

Theorem. 6 Main Problem
Following the procedure of *Lemma 4* and by *Corollary 5* we can keep constructing V_{p_1}, V_{p_2}, ..etc, which in turn are *at most countable.* There are two cases:

- (i) We reach some empty X_i, So $\{V_i\}$ is finite. Or

- (ii) We do not ever reach an empty X_i, and $\{V_i\}$ is *countable.*

Note. I received the following support before being able to solve the problem. I admit it was totally unlikely to think of the formulation $(q-\delta, q+\epsilon) \subset X$ on my own. I admit the problem is completely spoiled.



Available Today at 9:05 PM
I was insinuating that the construction of bigger and bigger intervals is a good idea.
Pick a rational in your open set. What's the largest interval you can put around it without leaving your set?

@Available Pick a rational in your open set. What's the largest interval you can put around it without leaving your set?
Mostafa Touny Today at 9:07 PM
I am not aware of any mathematical operation that generates a largest interval given an element of it

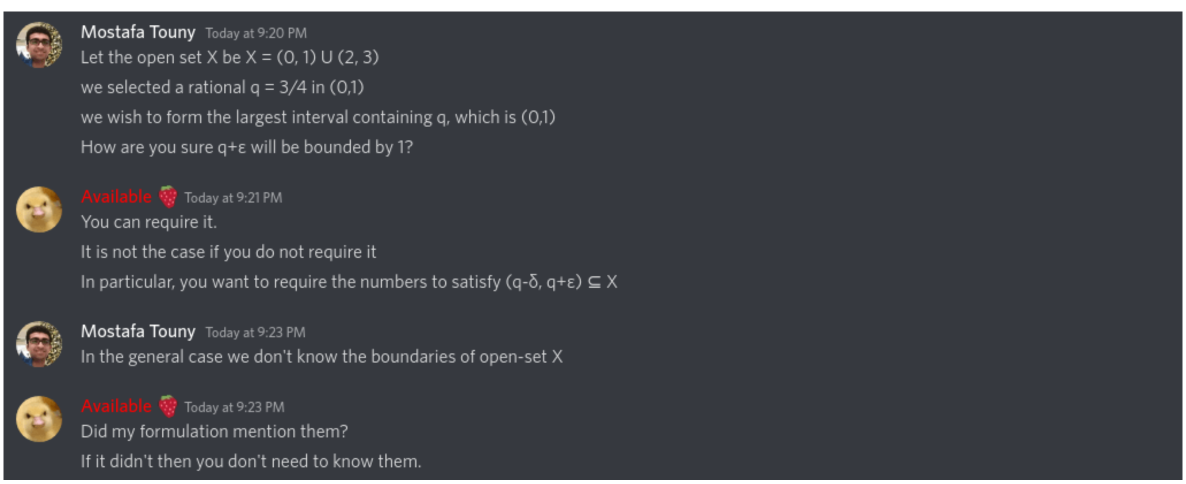
@Mostafa Touny I am not aware of any mathematical operation that generates a largest interval given an element of it
Available Today at 9:08 PM
Try to construct it with the operations you know

@Available Try to construct it with the operations you know
Mostafa Touny Today at 9:10 PM
Do I need to pick-up another rational number, or only one suffices?
Since every point of an open-set is interior, we know there's a segment (a_0, b_0) containing the rational number r_0.
Can we just define (a,b) to be the largest interval containing r_0?

@Mostafa Touny Do I need to pick-up another rational number, or only one suffices?
Available Today at 9:12 PM
If q is that rational number then you can put an interval around q by making it a bit smaller, say q-δ for some δ > 0 and a bit bigger, say q+ε for some ε > 0. That's all you need. Then you can use a set operation

@Available If q is that rational number then you can put an interval around q by making it a bit smaller, say q-δ for some δ > 0 and a bit bigger, say q+ε f...
Mostafa Touny Today at 9:16 PM
You mean forming a radius around q where numbers considered do fall within that radius? (edited)

Available Today at 9:16 PM
No, not a radius.
It's not going to have equal lengths on both sides



Mostafa Touny Today at 9:20 PM
Let the open set X be X = (0, 1) U (2, 3)
we selected a rational q = 3/4 in (0,1)
we wish to form the largest interval containing q, which is (0,1)
How are you sure q+ε will be bounded by 1?

Available Today at 9:21 PM
You can require it.
It is not the case if you do not require it
In particular, you want to require the numbers to satisfy (q-δ, q+ε) ⊆ X

Mostafa Touny Today at 9:23 PM
In the general case we don't know the boundaries of open-set X

Available Today at 9:23 PM
Did my formulation mention them?
If it didn't then you don't need to know them.

Mostafa Touny Today at 9:42 PM

> for p ≠ q we have either V_q = V_p or V_q ∩ V_p = ∅.

I guess that enables us to conclude X_i = X_i-1 - S_i-1 is either
- empty, or
- non-empty open-set

but never non-open. right?

Available 🍓 Today at 9:43 PM

I'm not sure how you want to conclude that X_i is open from that

Mostafa Touny Today at 9:43 PM

So you mean X_i might be a closed non-empty set?

Available 🍓 Today at 9:44 PM

No, I just don't see how the conclusion follows from this particular premise