Problem-Set 03

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Contents

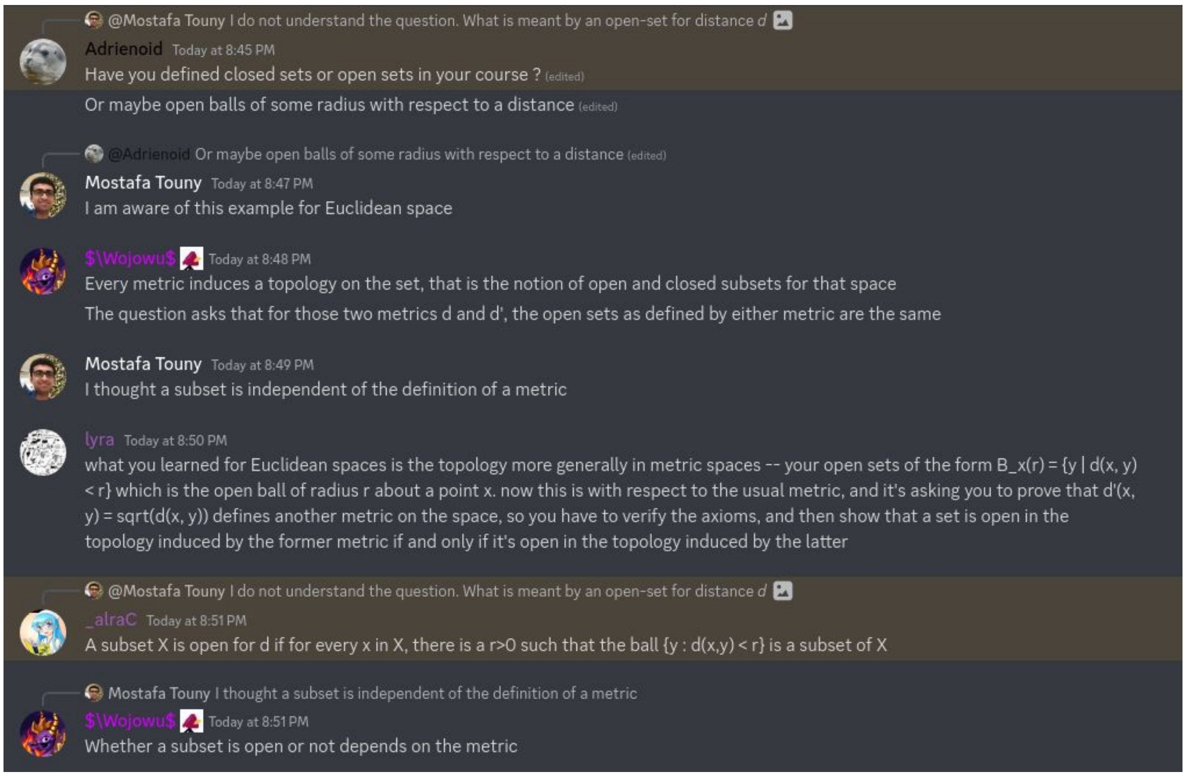
Problem 1

The required conditions follow naturally as:

- $d'(x, x) = \sqrt{d(x, x)} = \sqrt{0} = 0$.
- If $d(x, y) > 0$ then $d'(x, y) > 0$ as the square root of non-zero is non-zero. Otherwise $0^2 = 0$ contradicting the fact $d'(x, y) > 0$.
- $d'(x, y) = \sqrt{d(x, y)} = \sqrt{d(y, x)} = d'(y, x)$.
- $d'(x, y) = \sqrt{d(x, y)} \leq \sqrt{d(x, r) + d(r, y)} \leq \sqrt{d(x, r)} + \sqrt{d(r, y)} = d'(x, r) + d'(r, y)$.

For an arbitrary open-set of d, $\{y \mid d(x, y) < r\}$ there is an equivalent open-set of d', $\{y \mid d'(x, y) < \sqrt{r}\}$. For an arbitrary open-set of d', $\{y \mid d'(x, y) < r\}$, there is an equivalent open-set of d, $\{y \mid d(x, y) < r^2\}$.

Note. Some good friends assisted in solving this problem.



Problem 2

Lemma. For any point p in R, There exists a smallest element in the set $\{q \in E \mid q > p\}$

Assume to the contrary that no smallest element exists. Then as the set is bounded below, the *infimum* exists, and is a limit point. That contradicts our hypothesis of no limit points in E.

Corollary. $E \cap R^+ = E^+$ has a smallest element

By the above lemma set $p = 0$.

Corollarly. Given $x_i \in E^+$ there exists a smallest element among $E^+ \cap \{y \mid y > x_i\}$

By the above lemma set $p = x_i$.

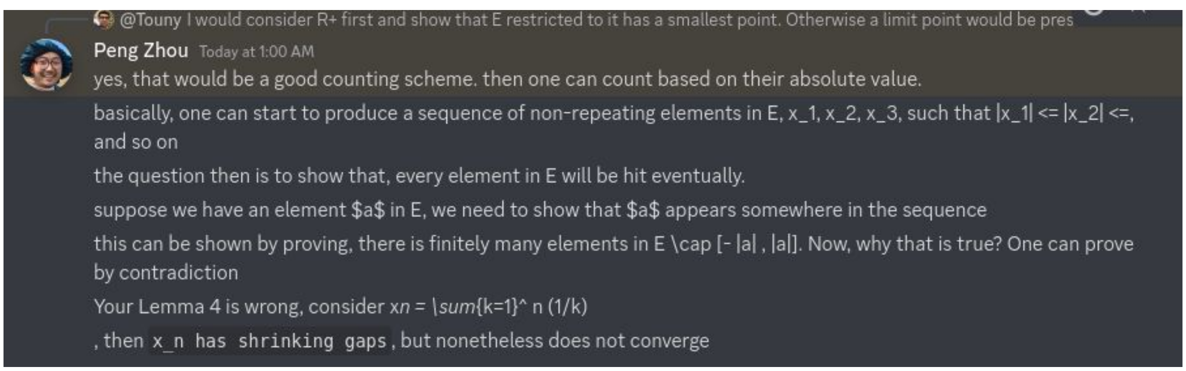
Now we have a counting scheme on E^+. What is remaining now is to prove every element in E will be hit eventually. The following lemma suffices.

Lemma. there are finitely many elements in $E \cap [-|a|, |a|]$

Assuming the contrary for the sake of contradiction, We get infinite elements in $E \cap [-|a|, |a|]$. Those are present in both E and $[-|a|, |a|]$ by definition. Since $[-|a|, |a|]$ is compact we know any infinite subset has a limit point (*Theorem 2.41, p. 40* in *baby-rudin*). But then we get a limit point in E. Contradiction

Similarly we can prove $E \cap R^- = E^-$ is countable, and hence E is countable also.

Note. 1 Professor Peng Zhou hinted the solution approach



Note. 2 Through chatting with good friends a cleaner alternative proof can be made as, "Because E has no limit points it is closed. Assume E is uncountable. Then there is an integer n such that intersection with [n,n+1] is also uncountable. This intersection is closed and bounded, thus compact. So we can take a sequence inside this intersection and it will have a convergent subsequence contradicting the assumption on limit points"

Mostafa Touny Yesterday at 11:09 PM

I conjecture the following approach: Establish an enumeration process of sequence x_i in E, And prove there is a discrete minimum distance from x_i to x_i+1 .

> 2. Consider $\mathbb{R}$ with the standard metric. Let $E \subset \mathbb{R}$ be a subset which has no limit points. Show that E is at most countable. (3 points)

Even if my approach is correct, I feel the proof is going to be complicated, and that there's a cleaner way.

Do you think the approach I articulated is a good one or tedious as I guessed?

Crazy Carla Yesterday at 11:12 PM

Why are you assuming that E is countable?

Poopheeler II: Wrath of Khanway Yesterday at 11:12 PM

Do you need to assume E is countable to do the enumeration x_i to begin with?

^

Mostafa Touny Yesterday at 11:12 PM

No

I would consider R+ first and show that E restricted to it has a smallest point. Otherwise a limit point would be present.

I guess my technique is clear now

December 6, 2022

FShrike on MSE Today at 12:13 AM

If I'm not mistaken, a set with no limit points is necessarily discrete (in any Hausdorff space) and the only discrete subsets of R are countable

Gal(Q(ζ_𝒜)/Q) Today at 1:22 AM

I think there's a cute way using Heine-Borel (edited)

Gal(Q(ζ_𝒜)/Q) Today at 1:34 AM

Because E has no limit points it is closed. Assume E is uncountable. Then there is an integer n such that intersection with [n,n+1] is also uncountable. This intersection is closed and bounded, thus compact. So we can take a sequence inside this intersection and it will have a convergent subsequence contradicting the assumption on limit points

> @Gal(Q(ζ_𝒜)/Q) Because E has no limit points it is closed. Assume E is uncountable. Then there is an integer n such that intersection with [n,n+1] is a...

geogristle Today at 3:52 AM

u gotta specify distinct elements of sequence

Available 🍓 Today at 4:53 AM

Suppose $n(x)=\inf{m\in\bN \mid |B(x, 1/m) \cap E| = 1}$ for $x\in E$. Then $n(x)\in\bN$ and ${B(x,1/n(x))}_{x\in E}$ is an open cover of E. Since $\bR$ is heredetarily Lindelöf, in the sense of the link I post, there is a countable subcover. However, since this cover consists of disjoint subsets of E that contain exactly one member of E, this countable subcover must be exactly the original cover and since E is in bijection with this cover, E must be countable.

𝒜olodøx ✓ BOT Today at 4:53 AM

Available

> Suppose $n(x) = \inf\{m \in \mathbb{N} \mid |B(x, 1/m) \cap E| = 1\}$ for $x \in E$. Then $n(x) \in \mathbb{N}$ and $\{B(x, 1/n(x))\}_{x\in E}$ is an open cover of E. Since $\mathbb{R}$ is heredetarily Lindelöf, in the sense of the link I post, there is a countable subcover. However, since this cover consists of disjoint subsets of E that contain exactly one member of E, this countable subcover must be exactly the original cover and since E is in bijection with this cover, E must be countable.

Available 🍓 Today at 4:54 AM

The link https://math.stackexchange.com/a/2320467/750710

> @Gal(Q(ζ_𝒜)/Q) Because E has no limit points it is closed. Assume E is uncountable. Then there is an integer n such that intersection with [n,n+1] is a...

Mostafa Touny Today at 8:53 AM

> E is uncountable. Then there is an integer n such that intersection with [n,n+1] is also uncountable

Would you recommend me a resource for this?

Poopheeler II: Wrath of Khanway Today at 8:55 AM

Assume the negation. Then E is the union of disjoint countable sets $E\cap [n,n+1]$, and a countable union of countable sets is countable. But E is uncountable (edited)

👍 1

Problem 3

Assume for the sake of contradiction that the process does not stop after a finite number of steps. Then the sequence x_i is infinite. Consider the infinite subset $\{x_i\} = S_\delta$; By hypothesis it has a limit point in X, Call it p. So for neighbourhood $N_{\delta/4}(p)$, some point $q_1 \neq p$ is in that neighbourhood. Let $r_1 = d(p, q_1)$. Consider neighbourhood $N_{r_1/2}(p)$; Clearly q_1 is not in it. So there is a point $q_2 \neq q_1$ in it. We have now distinct points $q_1, q_2 \in S$ such that $d(p, q_1) \leq \delta/4$ and $d(p, q_2) \leq \delta/4$. It follows $d(q_1, q_2) \leq d(q_1, p) + d(p, q_2) \leq \delta/4 + \delta/4 = \delta/2$. But the construction of sequence x_i stipulates every pair of points is of distance at least δ. Contradiction.

It follows by the above lemma, that for any point p in X, the distance between it and some x_i of S is strictly less than δ. Therefore p is covered by $N_\delta x_i$.

Now we prove X is separable. We know for each $\delta = 1/n$, The corresponding subset $S_{1/n}$ is finite. Clearly $\cup_n S_{1/n} = S$ is countably infinite. It suffices to show, For a point $p \in X - S$, it can get arbitrarily close to points of S. Consider arbitrary $\delta > 0$ and its corresponding neighbourhood $N_\delta(p)$.

Take $\delta' = \delta/2$, and $n' > 0$ such that $1/n' < \delta'$. Consider $N_{\delta'}(p)$. There are two cases.
Case 1: A point $q \in S_{1/n'}$ is in $N_{\delta'}(p)$, Then it is also in $N_\delta(p)$.
Case 2: No point $q \in S_{1/n'}$ is in $N_{\delta'}(p)$. Then for any $z \in N_{\delta'}(p)$ some point $q \in S_{1/n'}$ exists such that $d(z, q) < 1/n'$. It follows $\delta = \delta/2 + \delta/2 > \delta' + 1/n' > d(p, z) + d(z, q) \geq d(p, q)$. In other words, $q \in N_\delta(p)$.

Problem 4

Proposition 1. The distance function $d : X \times X \to R$ in a metric space X is continuous.

Proof. Fix (a, b). Let $\epsilon > 0$. We can take small enough δ such that $d(a, x) < \epsilon/2$ and $d(b, y) < \epsilon/2$. By the triangular inequality $d(x, y) \leq d(x, a) + d(a, b) + d(b, y)$. Hence $|d(x, y) - d(a, b)| < |d(x, a) + d(b, y)| < \epsilon$.

Proposition 2. The function $g(x) = d(x, f(x))$ is continuous over X.

Proof. Define a vector-valued function $h(x) = (h_1(x), h_2(x))$ where $h_1(x) = x$ is the identity and $h_2(x) = f(x)$. Then h is continuous, and so is the composite function $g = d \circ h$.

Theorem. Problem statement.

As before let $g : X \to \mathbb{R}$ by $x \mapsto d(x, f(x))$. The image $\{d(x, f(x)) \mid x \in X\}$ is lower-bounded by 0. Since X is non-empty and $\mathbb{R}$ has the *greatest-lower-bound property*, It follows $inf\, X = m$ exists. Assume for contradiction $m > 0$. By *thm 4.16, p 89, rudin*, there is a point $p_0 \in X$ where $g(p_0) = m = d(p_0, f(p_0))$. But we are given

$d(f(p_0), f^2(p_0)) < m$, i.e $g(f(p_0)) < g(p_0)$. Contradiction. Therefore $inf\, X = 0$ and there's a point p such that $g(p) = d(p, f(p)) = 0$, implying $f(p) = p$.