Problem-Set 04

Mostafa Touny

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Contents

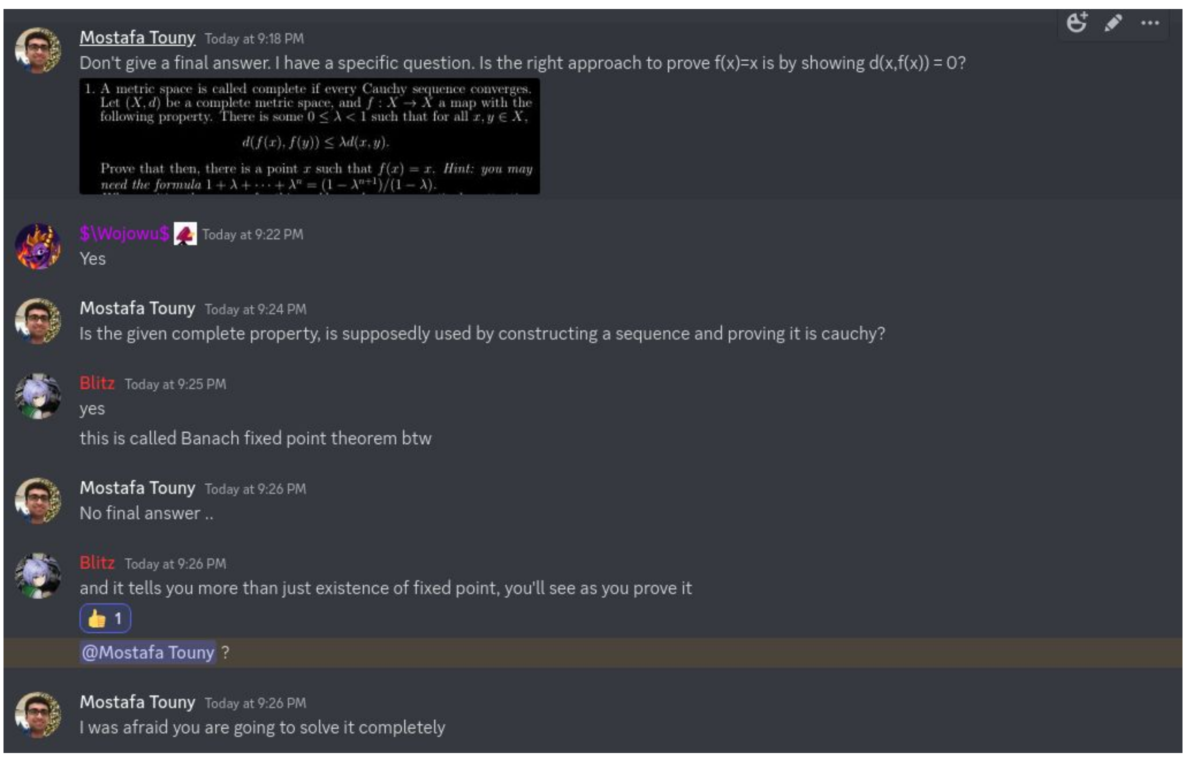
Problem. 1

Lemma. 1 If $x_{n+1} \leq \lambda x_n$, where $0 \leq \lambda < 1$, Then the sequence $\{x_n\}$ gets artbitrarily small
Clearly $x_{1+k} \leq \lambda^k x_1$, by substituting successive terms in the inequality. Given $\epsilon > 0$ we can reach $\lambda^k x \leq \epsilon$ by setting $k \geq \log_\lambda y/x$.

Fix any x in the metric space, Then construct the following sequence: $\{f^n(x)\} = f^0(x), f^1(x), f^2(x), \ldots$. We prove it is cauchy. Consider $d(f^n(x), f^m(x))$ of some tail where $n < m$. By the *triangular inequality*, We know the distance is upper-bounded by $d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^{n+2}(x)) + \cdots + d(f^{m-1}(x), f^m(x)) \leq (m - n + 1)\ \lambda^{n-1}\ d(f^1(x), f^2(x))$. By *Lemma 1* and substituting distances by a sequence $\{x_n\}$ our intended result is concluded. ■

Given X is *complete* we know our sequence $\{f^n(x)\}$ converges. Call it q. We show it converges also to $f(q)$, and by the uniqueness of limits, The main theorem of $f(x) = x$ for some x is concluded. Observe $d(f^{n+1}(x), f(q)) \leq d(f^n(x), q)$, but the right hand side of the inequality is arbitrarily small. ■.

Note. This problem was solved with assistance by wonderful friends. The main key idea of using the uniqueness of limits was given by them. See the following chat:



Blitz Today at 9:26 PM
no, I saw the original message
1

$\Wojowu$ Today at 9:29 PM
I will only give initial answer

Mostafa Touny Today at 9:34 PM
Does the proof use continuous functions as a technique?

Blitz Today at 9:34 PM
not directly, no

Mostafa Touny Today at 9:44 PM
Fixing any x, Is the sequence x, f^1(x), f^2(x), ... right?

Blitz Today at 9:44 PM
yeah

Mostafa Touny Today at 9:50 PM
Is combining
d(f^2(x), f^1(x)) <= \lambda d(f^1(x), f^0(x)), and
d(f^3(x), f^2(x)) <= \lambda d(f^2(x), f^1(x))
To conclude d(f^3(x), f^2(x)) <= \lambda^2 d(f^1(x), f^0(x))
A right step? (edited)

@Mostafa Touny Is combining d(f^2(x), f^1(x)) <= \lambda d(f^1(x), f^0(x)), and d(f^3(x), f^2(x)) <= \lambda d(f^2(x), f^1(x)) To conclude d(f^3(x), f^2
Blitz Today at 9:51 PM
last inequality is wrong
but yes that's the idea

Mostafa Touny Today at 9:51 PM
typo, fixed it. (edited)

Blitz Today at 9:51 PM
but remember you'll have to show it's Cauchy

@Blitz but remember you'll have to show it's Cauchy
Mostafa Touny Today at 9:52 PM
It is on my agenda
Is $d(f^3(x),f^2(x)) + d(f^2(x), f^1(x)) \leq (\lambda + \lambda^2) d(f^1(x), f^0(x))$ a right step? (edited)

Ꮇolodøx ✓ BOT Today at 9:54 PM
Mostafa Touny
Is $d(f^3(x), f^2(x)) + d(f^2(x), f^1(x)) \leq (\lambda + \lambda^2)d(f^1(x), f^0(x))$ a right step?

Blitz Today at 9:56 PM
in some sense, yes

Mostafa Touny Today at 9:58 PM
and I guess we then obtain the general form
$d(f^{k+1}(x), f^1(x)) \leq (\lambda + \lambda^2 + \dots + \lambda^k) d(f^1(x),f^0(x))$

Ꮇolodøx ✓ BOT Today at 9:58 PM
Mostafa Touny
and I guess we then obtain the general form $d(f^{k+1}(x), f^1(x)) \leq (\lambda + \lambda^2 + \cdots + \lambda^k)d(f^1(x), f^0(x))$

Mostafa Touny Today at 9:58 PM
That is the only way I can think of using the given hint

FShrike on MSE Today at 9:59 PM
the given hint isn't the usual way of doing this
👍 1

> Mostafa Touny Is combining d(f^2(x), f^1(x)) <= \lambda d(f^1(x), f^0(x)), and d(f^3(x), f^2(x)) <= \lambda d(f^2(x), f^1(x)) To conclude d(f^3(x), f^2(x

Mostafa Touny Today at 9:59 PM
So we return to focus here

> @Mostafa Touny Is combining d(f^2(x), f^1(x)) <= \lambda d(f^1(x), f^0(x)), and d(f^3(x), f^2(x)) <= \lambda d(f^2(x), f^1(x)) To conclude d(f^3(x), f^2

FShrike on MSE Today at 10:00 PM
I think you meant *lambda* for λ (edited)

𝒥olodøx ✓ BOT Today at 10:00 PM
FShrike on MSE

I think you meant *lambda* for λ

Mostafa Touny Today at 10:00 PM
You are right
fixed it

> @FShrike on MSE the given hint isn't the usual way of doing this

Blitz Today at 10:01 PM
huh? Pretty sure it is

> ✓ BOT @𝒥olodøx **Mostafa Touny**

Blitz Today at 10:03 PM
you want a bound for $d(f^n(x), f^m(x))$

𝒥olodøx ✓ BOT Today at 10:03 PM
Blitz

you want a bound for $d(f^n(x), f^m(x))$

Mostafa Touny Today at 10:03 PM
My problem is we can have a sequence of ${ d(f^{n+1}(x), f^{n}(x)) }$ converging to zero, but no *distance* ever reaches exactly zero.
(edited)

𝒥olodøx ✓ BOT Today at 10:04 PM
Mostafa Touny
Compile Error! Click the ⚠ reaction for more information.
(You may edit your message to recompile.)

My problem is we can have a sequence of $\{d(f^{n+1}(x), f^n(x))\}$ converging to zero, but no $_distance_everreachesexactlyzero.$

> @Blitz you want a bound for $d(f^n(x), f^m(x))$

Mostafa Touny Today at 10:04 PM
Same intuition, A bound may arbitrarily get the distance close to zero but never equal to it.
I do not see how a bound can be useful to obtain d(f(x), x) = 0

> @Mostafa Touny Same intuition, A bound may arbitrarily get the distance close to zero but never equal to it.

Blitz Today at 10:05 PM
you want to show this sequence is Cauchy

> @Blitz huh? Pretty sure it is

FShrike on MSE Today at 10:05 PM
Sorry, you can use it [redacted]. It's just not the first step, I got confused with the intentions

@Mostafa Touny I do not see how a bound can be useful to obtain d(f(x), x) = 0

Blitz Today at 10:05 PM

it won't have to be the same x

and often won't be

Mostafa Touny Today at 10:06 PM

You are right

@Blitz you want to show this sequence is Cauchy

Mostafa Touny Today at 10:07 PM

For it to be cauchy we must bound the distance on any f^n(x), f^m(x) as you said, of the tail of course (edited)

so that the diameter is bounded

FShrike on MSE Today at 10:07 PM

A nice follow up q. If the metric space X is compact, and f:X->X satisfies d(f(x),f(y))<d(x,y) for all x and y, f has a fixed point

@FShrike on MSE A nice follow up q. If the metric space X is compact, and f:X->X satisfies d(f(x),f(y))<d(x,y) for all x and y, f has a fixed point

Mostafa Touny Today at 10:08 PM

Thank you for the note. Let's investigate that after finishing this problem.

@FShrike on MSE A nice follow up q. If the metric space X is compact, and f:X->X satisfies d(f(x),f(y))<d(x,y) for all x and y, f has a fixed point

Blitz Today at 10:08 PM

x =/= y

Mostafa Touny Today at 10:09 PM

Without the loss of generality we can assume n > m.

It is proven using triangle inequality of distances, right?

Blitz Today at 10:09 PM

yes

Mostafa Touny Today at 10:10 PM

Before working out a careful proof, Even if I proved this sequence is cauchy, I do not see how to obtain some x where d(f(x), x) = 0. (edited)

Do we need to use any special toolbox?

like a theorem from Rudin's book?

$\Wojowu$ Today at 10:11 PM

Recall that X is complete

Mostafa Touny Today at 10:12 PM

Then the sequence converges, call it q.

Distances decreasingly converge to q. I don't see any insight

Assuming d(f(x), x) > 0 does not yield any contradiction

Blitz Today at 10:15 PM

f^(n+1)(x) = f(f^n(x)) while f^n(x) and f^(n+1)(x) converge to the same point

here we use continuity

$\Wojowu$ Today at 10:18 PM

You don't need to explicitly use continuity

Mostafa Touny Today at 10:18 PM

LEC #	TOPICS	READINGS
1	Sets, ordered sets, countable sets	pp. 3–6, and 24–30; Theorem 2.14 makes good reading even though it wasn't covered in class.
2	Fields, ordered fields, least upper bounds, the real numbers	pp. 5–11
3	The Archimedean principle; decimal expansion; intersections of closed intervals; complex numbers, Cauchy-Schwarz	pp. 38–9 (Theorem 2.38 only), 41 (Corollary to Theorem 2.43 only), and 12–6 **Independent Reading:** pp. 16–7 (Euclidean spaces)
4	Metric spaces, ball neighborhoods, open subsets	pp. 30–4
5	Open subsets, limit points, closed subsets, dense subsets	pp. 34–46
6	Compact subsets of metric spaces	pp. 36–8, and Problem 26 from p. 45
7	Limit points and compactness; compactness of closed	pp. 38–40

$\Wojowu$ Today at 10:18 PM
You can instead try showing that the sequence f^n(x) converges to f(x)

> Mostafa Touny The problem is up to lecture 9

Mostafa Touny Today at 10:18 PM
problem-set 04 https://ocw.mit.edu/courses/18-100c-real-analysis-fall-2012/pages/assignments/

> @$\Wojowu$ You can instead try showing that the sequence f^n(x) converges to f(x)

Mostafa Touny Today at 10:22 PM
f(x) is in the sequence { f^n(x) }_n. If the convergence you mentioned is true then some tail range consists of only f(x).
Am I thinking on the right direction?

$\Wojowu$ Today at 10:23 PM
Sorry, I meant f^n(x) converge to f(q)

Mostafa Touny Today at 10:24 PM
Since the limit is unique, it immediately follows q = f(q)

$\Wojowu$ Today at 10:25 PM
Right, but you need to prove f(q) is also the limit

Mostafa Touny Today at 10:25 PM
Yes
$d(f^{n+1}(x), f(q)) \leq d(f^n(x), q)$

Ꭾolodøx ✓ BOT Today at 10:31 PM
Mostafa Touny

$$d(f^{n+1}(x), f(q)) \leq d(f^n(x), q)$$

Mostafa Touny Today at 10:31 PM
But we can set the right hand side arbitrarily small. Done
A critical point of discussion is the follows. What is the insight behind guessing the uniqueness of limit as the proof technique?

$\Wojowu$ Today at 10:34 PM
Well, the insight we have is that f has to be continuous, so f^n(x) converging to q implies f^{n+1}(x) converges to f(q)

FShrike on MSE Today at 10:34 PM
You wanted f(x)=x. You knew f contracted the space, you could visualise successive iterates of f as collapsing all of X onto some point. That point has to be fixed

$\Wojowu$ Today at 10:34 PM
If you know what continuity is, this would be obvious. If you don't you have to strip the argument down a bit

Mostafa Touny Today at 10:35 PM
Can you imagine an insight which does not rely on continuity as a background toolbox?

$\Wojowu$ Today at 10:36 PM
You probably can

Mostafa Touny Today at 10:39 PM
FIrst. It seems impractical to go through every toolbox in Rudin, and imagine an argument using it. A motivating insight of the problem must motivate the search for toolboxes.
Second. Maybe I could have imagined the toolbox d(f(x),f(y)) <= d(x,y) applied on f^n(x) converging to q, but that seems too faraway from my intuition and seems very close to the notion of continuity. I won't rely on it.
Correct me if you disagree, please

$\Wojowu$ Today at 10:42 PM

Well it's really hard to fairly answer such questions. If you know a solution (and came up with it using continuity say like I did) then you can bake up a dozen vaguely plausible explanations of how one could arrive at it. But you can't ever know what is the "intended" line of reasoning or whether there is one like that at all

Mostafa Touny Today at 10:43 PM

I meant, whether we can come up with a new line of reasoning, other than the one of continuity

and yet, reach the same key insight, of using limit uniqueness as the proof technique.

$\Wojowu$ Today at 10:43 PM

Same thing as I said applies

I came up with the new line of reasoning by thinking about continuity

FShrike on MSE Today at 10:45 PM

What's wrong with "being close to continuity"?

continuity is a very essential concept

Mostafa Touny Today at 10:45 PM

Thank you @$\Wojowu$ for the insightful discussion. I promise to contribute to this community at some day.

𝒯olodøx ✓ BOT Today at 10:45 PM

Mostafa Touny

Thank you @Whoever pinged is a cutie! for the insightful discussion. I promise to contribute to this community at some day.

$\Wojowu$ Today at 10:46 PM

You're already contributing

2

Mostafa Touny Today at 10:46 PM

@FShrike on MSE, but as someone who takes the course for the first time, I am expected to solve the problem without having continuity as a background-insight.

FShrike on MSE Today at 10:47 PM

Really? Sequences, limits, metric spaces, convergence,... continuity should be on that course!!

$\Wojowu$ Today at 10:47 PM

I don't think there is any problem with having continuity as an insight

Mostafa Touny *Click to see attachment*

Mostafa Touny Today at 10:47 PM

The problem is up to lecture 9

$\Wojowu$ Today at 10:47 PM

Even if you are expected to not use it in your proof, no issue using it as motivation

FShrike on MSE Today at 10:47 PM

The argument is all about using 'closeness' ideas. That's more or less what continuity is about for metric spaces

Mostafa Touny Today at 10:48 PM

So you mean, taking continuity insight from calculus, and using that insight here to come-up with the proof idea?

FShrike on MSE Today at 10:48 PM

meh not from calculus

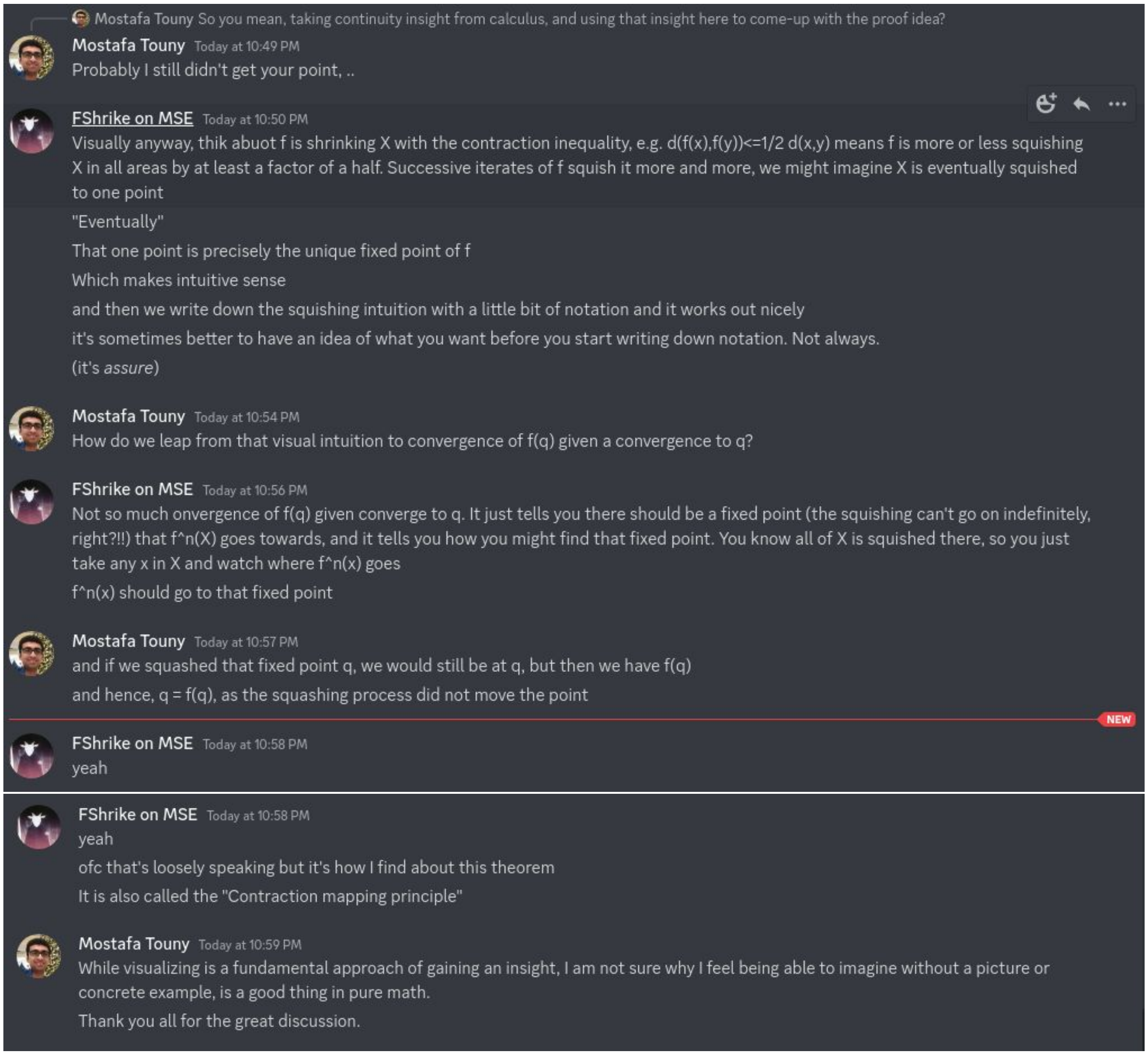
@FShrike on MSE The argument is all about using 'closeness' ideas. That's more or less what continuity is about for metric spaces

Mostafa Touny Today at 10:48 PM

I ensure you up to this problem, continuity is not covered. See the course from here: https://ocw.mit.edu/courses/18-100c-real-analysis-fall-2012/pages/readings/

FShrike on MSE Today at 10:49 PM

Sure. I just find that surprising



Problem. 2

Suppose (x_k) converges to q. Let $\epsilon > 0$ be arbitrary. We already have N_0 where for any $k \geq N_0$ $x_k - q < \epsilon$. For a given permuted sequence $(x_{g(k)})$, We now show there's N_1 where for any $n \geq N_1$, $x'_n - q < \epsilon$.

Observe $x_1, \ldots, x_{N_0-1}$ are finite. Consider indices $g(1), \ldots, g(N_0 - 1)$ and take the maximum. Call it $g_{max}(N_0 - 1)$. Clearly for any index i greater than it, we know x'_i is not equal to any one of $x_1, \ldots, x_{N_0-1}$. So it is contained in the trail $x_{N_0}, x_{N_0+1}, \ldots$. Thus, $x'_i - q < \epsilon$ for any $i > g_{max}(N_0 - 1)$.

It is not true if we dropped the assumption that g is one-to-one. A counter example is a permutation function whose range is exactly one element of $\mathcal{N}$.

Problem. 3

The is exactly the same as theorem 3.4 in Rudin's page 50.

Problem. 4

Lemma. $v_p(p^k + p^{k+1} + \cdots + p^{k+m}) = k$.

Observe $p^k + p^{k+1} + \cdots + p^{k+m} = p^k(1 + p^1 + \cdots + p^m)$. Moreover $p \nmid (1 + p^1 + \cdots + p^m)$ as $p \mid (p^1 + \cdots + p^m)$. It holds if $m = 0$. ■

Theorem. $(x_i) = \sum_{j=0}^{i} p^i$ is a cauchy sequence.

Let $\epsilon > 0$ be arbitrary. By the Archimedean property $\exists N, 1/N < \epsilon$. Set $H_\epsilon = N$ and $n, m \geq H_\epsilon$.

if $n = m$, then $d(x_n, x_m) = 0 \leq \epsilon$.

WLOG assume $n > m$. It follows

$$d(x_n, x_m) = \left| \sum_{j=0}^{n} p^j - \sum_{j=0}^{m} p^j \right|_p = \left| \sum_{i=m+1}^{n} p^i \right|_p$$

$$v_p\left(\sum_{j=m+1}^{n} p^j \right) = v_p(p^{m+1} + p^{m+2} + \cdots + p^n) = m + 1$$

Hence $d(x_n, x_m) = p^{-(m+1)} \leq \dfrac{1}{m+1} \leq \dfrac{1}{N} < \epsilon$, for all $n, m \geq N$.

Theorem. Convergence when $p = 2$.

Observe $(x_i) = \sum_{j=0}^{i} 2^i = 2^{i+1} - 1, \forall i \geq 0$.

Then $d(x_i, -1) = d(2^{i+1} - 1, -1) = |2^{i+1} - 1 - (-1)|_2 = |2^{i+1}|_2 = 2^{-(i+1)}$.

Now as $i \to \infty$, $d(2^{i+1} - 1, -1) \to 0$, i.e $\lim_{i \to \infty} 2^{i+1} - 1 = -1$.

A more careful proof. set $\epsilon > 0$, then by the Archimedean property $\exists N, 1/N < \epsilon$. Set $H_\epsilon = N$. Then for $i \geq H_\epsilon$, $2^{-(i+1)} \leq 1/N \leq \epsilon$. ■