Problem-Set 08

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Problem. 1

By boundedness we get $|f(x)| \leq M_f$ and $|g(x)| \leq M_g$. Clearly there is N_g such that for $n \geq N_g$, $|g_n(x)| \leq M_g + \epsilon_g$.

Let $\epsilon > 0$ be arbitrary. Define $\epsilon_0 = \dfrac{\epsilon}{2(M_g + \epsilon_g)}$ and $\epsilon_1 = \dfrac{\epsilon}{2(M_f)}$.

By hypothesis we have can take N_{max} considering also N_g to get

$$\begin{aligned} |f_n(x) - f(x)| &< +\epsilon_0 \\ |g_n(x) - g(x)| &< +\epsilon_1 \end{aligned}$$

By multiplication,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g_n(x)| &< +\epsilon_0 \cdot |g_n(x)| \\ |f(x)g_n(x) - f(x)g(x)| &< +\epsilon_1 \cdot |f(x)| \end{aligned}$$

Now observe:

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\ &\leq |f_n(x)g_n(x) - f(x)g_n(x)| + |f(x)g_n(x) - f(x)g(x)| \\ &< \epsilon_0|g_n(x)| + \epsilon_1|f(x)| \\ &\leq \epsilon_0(M_g + \epsilon_g) + \epsilon_1(M_f) \\ &= \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

The second line follows by triangular inequality.

Problem. 2

Lemma. $\hat{f}$ is of the same class.

By definition, The domain of $\hat{f}$ is the same as f. Clearly $\hat{f}(0) = \frac{1}{4}f(2 \cdot 0) = \frac{1}{4}f(0) = \frac{1}{4}(0) = 0$ and $\hat{f}(1) = \frac{3}{4}f(2 \cdot 1 - 1) + \frac{1}{4} = \frac{3}{4}f(1) + \frac{1}{4} = \frac{3}{4}(1) + \frac{1}{4} = 1$.

The continuity of $\hat{f}$ follows by the continuity of f. Consider arbitrary $\hat{f}(q)$ and $\epsilon > 0$. Consider the case of $\hat{f}(q) = \frac{3}{4}f(2q-1) + \frac{1}{4}$ and note the other case is symmetric. Take $\grave{\epsilon} = \frac{4}{3}\epsilon$. By continuity of f, There exists δ such that for any r, if $|r - p| < \delta$ then

$|f(r) - f(p)| < \grave{\epsilon}$. Define $\grave{\delta} = \frac{\delta}{2}$, and observe for any r:

$$\begin{aligned}
\text{If} \quad & |r - q| < \grave{\delta} = \frac{\delta}{2} \\
\text{Then} \quad & |(2r+1) - (2q+1)| < \delta \\
\text{By Continuity} \quad & |f(2r+1) - f(2q+1)| < \grave{\epsilon} = \frac{4}{3}\epsilon \\
\text{Then} \quad & |(\frac{3}{4}f(2r+1) + \frac{1}{4}) - (\frac{3}{4}f(2q+1) + \frac{1}{4})| < \epsilon \\
\text{By definition} \quad & |\hat{f}(r) - \hat{f}(q)| < \epsilon
\end{aligned}$$

Lemma. $d(\hat{f}, \hat{g}) \leq \frac{3}{4}d(f, g)$.

If $x \geq \frac{1}{2}$,

$$\begin{aligned}
|\hat{f}(x) - \hat{g}(x)| &= |(\frac{3}{4}f(2x-1) + \frac{1}{4}) - (\frac{3}{4}g(2x-1) + \frac{1}{4})| \\
&= |\frac{3}{4}f(2x-1) - \frac{3}{4}g(2x-1)| \\
&= |\frac{3}{4}f(y) - \frac{3}{4}g(y)| \\
&= \frac{3}{4}|f(y) - g(y)|
\end{aligned}$$

where we define $y = 2x - 1$.

Since $|\hat{f} - \hat{g}|$ is defined in terms of $|f - g|$, Observe the maximum of $|f - g|$ yields the maximum of $|\hat{f} - \hat{g}|$.

Lemma. Exactly one f where $\hat{f} = f$.

Assume we have $\hat{f} = f$ and $\hat{g} = g$. By the previous lemma, $d(f, g) = d(\hat{f}, \hat{g}) \leq \frac{3}{4}d(f, g)$. This is true only if $d(f, g) = 0$ which concludes $|f(x) - g(x)| = 0$ for all x. In other words, $f = g$.

Problem. 3

We use the following theorem found in Rudin's book in page 59.

3.22 Theorem *Σa_n converges if and only if for every $\varepsilon > 0$ there is an integer N such that*

(6) $$\left|\sum_{k=n}^{m} a_k\right| \leq \varepsilon$$

if $m \geq n \geq N$.

Fix $x \in [a, b]$. The theorem follows by the following lemmas

- (i) $\sum_{k=n}^{m} f_k(x) \geq 0$ for odd n.
- (ii) $\sum_{k=n}^{m} f_k(x) \leq 0$ for even n.
- (iii) Given $f_n(x) = +M$ for odd n and non-negative M, $\sum_{k=n}^{m} f_n(x) \leq M$.
- (iv) Given $f_n(x) = -M$ for even n and non-negative M, $\sum_{k=n}^{m} f_n(x) \geq -M$.

Proof.

(i). Follows by a strong form of induction. Observe for odd n, $|f_n(x)| \geq |f_{n+1}(x)|$ yields $f_n(x) + f_{n+1} \geq 0$. The induction step is to show $\sum_{k=n}^{m+2} f_k(x) \geq 0$ given $\sum_{k=n}^{m} f_k(x) \geq 0$ and $\sum_{k=n}^{m+1} f_k(x) \geq 0$.

(ii). Symmetric to (i).

(iii) Expand to $f_n(x) + \sum_{k=n+1}^{m} f_k(x)$, Then it follows immediately by (ii)

(iv). Symmetric to (iii).

Theorem. These lemmas conclude, Given $f_n(x) = M$ regardless n is even or odd, $|\sum_{k=n}^{m} f_k(x)| \leq |M|$. But we are given $f_n(x)$ converges to 0, So we can substitute M by any $\epsilon > 0$.