Chapter 03

Mostafa Touny

September 17, 2023

Contents

Problems **2**
2 2
4 2
6(b) 2
7 2
10 3
19 3
30 3
34 3

Problems

2

$(Q, +)$. $\{\frac{x}{2} \mid x \in \mathcal{Z}\}$.

$(Q^*, *)$. $\{2^x \mid x \in \mathcal{Z}^+\} \cup \{\frac{1}{2^x} \mid x \in \mathcal{Z}^+\} \cup \{1\}$.

4

Consider $|x| = n$. Then $x^n = 1$ and no positive $r < n$ where $x^r = 1$. It follows

$$\begin{aligned} (x^n)^{-1} &= (1)^{-1} \\ (x \cdot x \cdot \dots \cdot x)^{-1} &= 1 \\ x^{-1} \cdot \dots \cdot x^{-1} &= \\ (x^{-1})^n &= \end{aligned}$$

Analogously if $(x^{-1})^r = 1$ then $x^r = 1$. That cannot happen for $r < n$.

6(b)

Identity is $e = 0$.

$|3| = 4$. $|8| = 3$. $|11| = 12$.

7

Fact. For any element x in any group, $x^{n+m} = x^n x^m$.

Fact. For any element x in any group, $(x^k)^m = x^{km}$.

$$\begin{aligned} (a^4 c^{-2} b^4)^{-1} &= (b^4)^{-1}(c^{-2})^{-1}(a^4)^{-1} \\ &= (b^4)^{-1}(c^2)(a^4)^{-1} \\ &= (b^7 b^{-3})^{-1}(c^2)(a^6 a^{-2})^{-1} \\ &= (b^{-3})^{-1}(c^2)(a^{-2})^{-1} \\ &= b^3 c^2 a^2 \end{aligned}$$

10

We naively construct all possible subgroups, pruning possible branches by their properties.

Any subgroup must have the identity element. $\{R_0\}$. (+1)

R_0, X is a subgroup for any reflection $X = H, V, D, D'$. (+4)

Considering a subgroup with R_0, X_0, X_1 for distinct reflections X_0, X_1 it must be the case we get rotation R_s for $s \neq 0$. So we cannot have a subgroup restricted on reflections other than the aforementioned case.

$\{R_0, R_{180}\}$. (+1)

For any subgroup with R_{90} or R_{270}, since it is closed it must contain also $\{R_0, R_{90}, R_{180}, R_{270}\}$. (+1)

For any subgroup containing $\{R_0, R_{180}, H\}$ it must contain also $\{R_0, R_{180}, H, V\}$. For any subgroup containing $\{R_0, R_{180}, V\}$ it must contain also $\{R_0, R_{180}, V, H\}$. (+1)

For any subgroup containing $\{R_0, R_{180}, D\}$ it must contain also $\{R_0, R_{180}, D, D'\}$. For any subgroup containing $\{R_0, R_{180}, D'\}$ it must contain also $\{R_0, R_{180}, D, D'\}$. (+1)

For any subgroup containing R_s for $s \neq 180$ and any reflection $X = H, V, D, D'$, since it is closed, it must contain also $\{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$. (+1)

So far we counted 10 subgroups.

19

We show the contrapositive. Assume $a^m = a^n$ for $m > n$. Then $a^m a^{-n} = a^n a^{-n}$ implying $a^{m-n} = e$, but $m - n > 0$ so a is of a finite order.

30

The question presumes the uniqueness of H. We won't prove it.

$H = \{2(9k_1 + 15k_2 + 20k_3) \mid k_1, k_2, k_3 \in \mathcal{Z}\}$.

Taking $k_1 = k_2 = k_3 = 0$ yields the identity $e = 0$. For $x \in H$ corresponding to k_i, Take $-(k_i)$ to obtain the inverse. Closed property is clear from the definition. Associativity follows from G. Odd numbers are excluded conforming to the fact H is a proper subgroup.

34

Since $e \in H$ and $e \in K$ by definition, We have $e \in H$.

if $x, y, z \in H \cap K$, then $x, y, z \in H$ and associativity follows.

if $x \in H \cap K$, then $-x \in H$ and $-x \in K$, and any element has an inverse.

if $x, y \in H \cap K$, then $x + y \in H$ and $x + y \in K$ by properties of a group.

A trivial argument by induction shows the intersection of any number of subgroups.