

Chapter 07

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Problems

1

Those are $\{a\langle 3 \rangle \mid a \in \mathcal{Z}\} = \{\{a \pm 0, a \pm 3, a \pm 6, \dots\} \mid a \in \mathcal{Z}\}$.

7

Observe $\langle a^4 \rangle = \{1, a^{4(1)}, a^{4(2)}, \dots, a^{4(14)}\}$. Then $|\langle a^4 \rangle| = 15$. It follows by *theorem 7.1* (page 142), The number of distinct left cosets is $30/15 = 2$.

9

Lazy to compute and typeset all left cosets.

H is a subgroup. Then by *theorem 7.1* (page 142), the number of left cosets of H in S_4 is $4!/4 = 3! = 6$.

10

Assume for contradiction $aH \cap bH = \phi$. Since we are given $aH = bK$, It follows $bK \cap bH = \phi$, Concluding $H \cap K = \phi$. Contradiction as the identity element $e \in G$ is common in both subgroups. Therefore $aH \cap bH \neq \phi$.

From *Lemma* (page 139), We get $aH = bH$. Then $bK = bH$ as we are given $aH = bK$. It follows $K = H$.

17

Let H be a proper subgroup of G . If $|H| = 1$, Then $H = \{e\} = \langle e \rangle$. it is cyclic. Now assume $|H| > 1$. Then by *theorem 7.1* (page 143), and without the loss of generality, $|H| = p$ for a prime p . By *corollary 3*, H is cyclic.

19

$5^{16} \pmod{7} = 6$ and $7^{13} \pmod{11} = 2$, Using the fact $ab \pmod{m} = (a \pmod{m})(b \pmod{m}) \pmod{m}$.

22

Let H and K be finite subgroups of a group G , Where $|H|$ and $|K|$ are coprime. Since $H \cap K$ is a subgroup of both H and K , By *theorem 7.1* (page 142), $|H \cap K| = 1$. Then $H \cap K = \{e\}$, where e is the identity of G .

38

39

We know all common divisors among 24 and 20 are 1, 2, 4. By *theorem 7.1* (page 142), It follows $|H \cap K| = 1, 2, \text{ or } 4$.

Case. $|H \cap K| = 1$. Then it is the trivial group of only the identity element.

Case. $|H \cap K| = 2$. Then it is $\{e, a\}$. Trivially abelian.

Fact. For any two elements a, b of a group. If $ab = b$ then $a = e$, the identity element. Observe we can cancel b in $ab = eb = b$.

Case. $|H \cap K| = 4$. Assume for contradiction, that $ab \neq ba$ for arbitrary distinct elements a and b , Neither of which is the identity. Then $ab \notin \{a, b\}$ by the *Fact*. Moreover $ab \neq e$ lest $b = a^{-1}$ and then $ab = ba$. Symmetrically these conclusions apply on ba . Since we excluded 3 elements out of 4, There is only a single element ab and ba can both be assigned to, i.e $ab = ba$. Contradiction.