Chapter 11

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Contents

Problems **2**
2 2
5 2
10 3
22 3
31 4

Problems

2

$n = 3$.

The table in page 213 shows that.

5

$45 = 3^2 \cdot 5^1$. By the *fundamental theorem of finite abelian groups*, All possible groups are

$$Z_9 \oplus Z_5 \approx Z_{45} \tag{1}$$
$$Z_3 \oplus Z_3 \oplus Z_5 \approx Z_3 \oplus Z_{15} \tag{2}$$

Group (1) has element 3 whose order is $|3| = 15$. Group (2) has element $(0, 1)$ whose order is $|(0, 1)| = 15$. Therefore, Any finite abelian group of order 45 has an element of order 15.

By *The fundamental theorem of cyclic groups* (page 81) we know all elements orders of Z_3 are: $1, 3$, and all elements orders of Z_{15} are: $1, 3, 5$. But by *Theorem 8.1* (page 158) all elements' orders of $Z_3 \oplus Z_{15}$ are: $1, 3, 5, 15$, by computing *lcm* of all possible pairs of elements orders. Therefore, It is not necessarily the case any finite abelian group of order 45 has an element of order 9.

10

$360 = 2^3 \cdot 3^2 \cdot 5^1$.

For 2^3, $k = 3$,

$$\begin{array}{rl} 3 & Z_8 \\ 2+1 & Z_4 \oplus Z_2 \\ 1+1+1 & Z_2 \oplus Z_2 \oplus Z_2 \end{array}$$

For 3^2, $k = 2$,

$$\begin{array}{rl} 2 & Z_9 \\ 1+1 & Z_3 \oplus Z_3 \end{array}$$

For 5^1, $k = 1$,

$$\begin{array}{rl} 1 & Z_5 \end{array}$$

It follows all groups are

$$
\begin{aligned}
&Z_8 \oplus Z_9 \oplus Z_5 \\
&Z_8 \oplus Z_3 \oplus Z_3 \oplus Z_5 \\
&Z_4 \oplus Z_2 \oplus Z_9 \oplus Z_5 \\
&Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_3 \oplus Z_5 \\
&Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_9 \oplus Z_5 \\
&Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_3 \oplus Z_5
\end{aligned}
$$

22

By *the fundamental theorem of finite abelian groups*, $G \approx Z_{p_1^{n_1}} \oplus Z_{p_2^{n_2}} \oplus \cdots \oplus Z_{p_k^{n_k}}$ where $|G| = p_1^{n_1} \cdot .. \cdot p_k^{n_k}$. We claim $n_1 = n_2 = \cdots = n_k = 1$.

Assume for contradiction some $n_i > 1$. Then by the theorem we can substitute $Z_{p_i^{n_i}}$ by $Z_{p_i^1} \oplus Z_{p_i^1} \oplus Z_{p_i^{n_i-2}}$. If $n_i = 2$ then just ignore the third term. It follows we have two distinct subgroups of cardinality p_i. In other words, two distinct subgroups of the same order of divisor p_i of $|G|$. Contradiction.

Therefore $G \approx Z_{p_1^1} \oplus Z_{p_2^1} \oplus \cdots \oplus Z_{p_k^1}$. But all p_is are coprime, So $G \approx Z_{p_1 \cdots p_k}$, Concluding it is cyclic.

31

If $a = b$ then $a^2 = b^2$. So a and b are distinct. Moreover $(a^2)^2 = a^4 = e$ and $(b^2)^2 = b^4 = e$. So we have distinct elements a^2 and b^2 of order 2.

By the *fundamental theorem of finite abelian groups*, All possible classes are:

$$
\begin{aligned}
&Z_{16} && (3) \\
&Z_8 \oplus Z_2 && (4) \\
&Z_4 \oplus Z_4 && (5) \\
&Z_4 \oplus Z_2 \oplus Z_2 && (6) \\
&Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 && (7)
\end{aligned}
$$

(3) is excluded as it has only one element of order 2, namely 8.

(4) is excluded. All orders of elements are $1, 2, 4, 8$ and $1, 2$ respectively. Elements of order 4 in group (4) can be only obtained by an element of order 4 in Z_8. Otherwise the *lcm* would be $1, 2, 8$. There are only two elements of order 4 in Z_8, namely 2 and 6. So all possible elements of order 4 in group (4) are $(2, 0), (6, 0), (2, 1), (6, 1)$. But the square of any of them is $(4, 0)$, Violating the given condition $a^2 \neq b^2$.

(6) is excluded. All orders of elements are $1, 2, 4$ and $1, 2$ respectively. There are only two elements in Z_8 of order 4, namely 1 and 3. So all possible elements of order 4 in group (4) are $(1, 0), (3, 0), (1, 1), (3, 1)$. But the square of any of them is $(2, 0)$, Violating the given condition of $a^2 \neq b^2$.

(7) is excluded as all elements orders of Z_2 are $1, 2$, So taking lcm would always be $1, 2$. So it has no element of order 4.

Therefore the class is group (5).