Chapter 12

Mostafa Touny

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Problems

2

It is 6. For any $i \in \{0, 2, 4, 6, 8\}$, $6i \mod 10 = i$.

3

It suffices to find a ring with a subgroup which in turn is not closed under multiplication. Particularly the ring of rationals $\mathbb{Q}$ and its subset $S = \{\frac{x}{2} \mid x \in \mathbb{Z}\} = \{x\ \frac{r}{2} \mid x \in \mathbb{Z}, y = 0, 1\}$. It is a subgroup as $\frac{x_0}{2} + \frac{x_1}{2} = \frac{x_0 + x_1}{2}$ where $x_0 + x_1 \in \mathbb{Z}$, and for $\frac{x_0}{2}$ there is $\frac{-x_0}{2}$ such that $\frac{x_0}{2} + \frac{-x_0}{2} = 0$. Observe $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \notin S$, So S is not closed under multiplication.

P.S.

- Any subgroup under addition of a ring, satisfies the ring's definition, except for being closed under multiplication.
- Any set S closed under usual addition of integers, is also closed under usual multiplication of integers, Since $ab = \underbrace{a + a + \cdots + a}_{b \text{ times}} \in S$.

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Unity's uniqueness. Let 1 and $1'$ be two unities. Then by definition $11' = 1'1 = 1'$, and $1'1 = 11' = 1$. So $1 = 1'$.

Multiplicative inverse uniqueness. Fix a_0. Let b_0 and b_1 be two multiplicative inverses of a_0. Then $b_0a_0 = a_0b_0 = 1$, and $b_1a_0 = a_0b_1 = 1$. So

$$\begin{aligned} a_0b_0 &= a_0b_1 \\ b_0(a_0b_0) &= b_0(a_0b_1) \\ (b_0a_0)b_0 &= (b_0a_0)b_1 \\ b_0 &= b_1 \end{aligned}$$

6

a. For Z_6, $3^2 = 3$ but $3 \neq 0$ and $3 \neq 1$.

b. For Z_4, $3 \cdot 3 = 0$ but $3 \neq 0$.

c. For Z_4, $2 \cdot 1 = 2 = 2 \cdot 3$ and $2 \neq 0$ but $1 \neq 3$.

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($\rightarrow$). By definition for some k,

$$\begin{aligned} bk &= c \\ bk \cdot 1 &= \\ bk \cdot aa^{-1} &= \\ ab \cdot ka^{-1} &= \end{aligned}$$

($\leftarrow$). By definition for some k,

$$\begin{aligned} ab \cdot k &= c \\ a \cdot bk &= \end{aligned}$$

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Consider arbitrary $ar_0a, ar_1a \in S$. Then

$$\begin{aligned} &ar_0a \cdot ar_1a \\ &\qquad = ar_0a^2r_1a \\ &\qquad = ar_0r_1a \in S \end{aligned}$$

As $r_0r_1 \in R$. Also,

$$\begin{aligned} &ar_0a - ar_1a \\ &\qquad = a[r_0a - r_1a] \\ &\qquad = a\left[(r_0 - r_1)a\right] \\ &\qquad = a(r_0 - r_1)a \in S \end{aligned}$$

As $r_0 - r_1 \in R$.

Since $1 \in R$, $a1a \in S$ but $a1a = a^2 = 1$.

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(1), (2), (3), (5), (6) of a ring's definition in page 227 are satisfied by the usual properties of matrix algebra and integers.

Note the additive identity is the matrix

$$\begin{bmatrix} 0 & 0+0 \\ 0+0 & 0 \end{bmatrix}$$

We show (4). For any matrix $M \in R$, where

$$M = \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$$

The matrix $-M$ defined as

$$-M = \begin{bmatrix} -a & -a+(-b) \\ -a+(-b) & -b \end{bmatrix}$$

is in $M_2(Z)$, as $-a \in Z$ whenever $a \in Z$. Clearly $M - M$ is the additive identity.

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$2 \in 2Z$ and $3 \in 3Z$ but $2 + 3 = 5 \notin 2Z \cup 3Z$, so $2Z \cup 3Z$ is not closed under addition.