Chapter 14

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Problems

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We use *Theorem 14.1* (ideal test) (page 249).

For $r_0 a, r_1 a \in \langle a \rangle$, We have $r_0 a - r_1 a = (r_0 - r_1)a \in \langle a \rangle$ by distributivity and $r_0 - r_1 \in R$.

For $r \in R$ and $r_0 a \in \langle a \rangle$, We have $r(r_0 a) = (rr_0)a \in \langle a \rangle$ by associativity and $rr_0 \in R$. Also $(r_0 a)r = r_0(ar) = r_0(ra) = (r_0 r)a$ by associativity and commutativity and $r_0 r \in R$.

3

The proof I is ideal by *Theorem 14.1* (ideal test) (page 249) is nearly identical to *Ex. 1*.

Let J be an arbitrary ideal that contains $a_1, a_2, \ldots, a_n$. Then by definition $ra_i \in J$. Since it's a group $r_1 a_1 + \cdots + r_n a_n \in J$ for any $r_i \in R$.

4

By the *subring test* (page 230), $S = \{(x, x) \mid x \in Z\}$ is a subring as $(x, x) - (y, y) = (x - y, x - y) \in S$ and $(x, x)(y, y) = (xy, xy) \in S$.

S is not an ideal as $(1, 1) \in S$ and $(1, 2) \in \mathbb{Z} \oplus \mathbb{Z}$ but $(1, 2)(1, 1) = (1, 2) \notin S$. In other words, $(1, 1)$ did not absorb $(1, 2)$.

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We use *Theorem 12.3* (subring test) (page 230). $(a+bi)-(a'+b'i) = (a-a')+(b-b')i \in S$ as $b - b'$ is even. $(a + bi)(a' + b'i) = (aa' - bb') + (ab' + a'b)i \in S$ as $ab' + a'b$ is even.

$1 + 2i \in S$ and $1 + 1i \in Z[i]$ but $(1 + 1i)(1 + 2i) = -1 + 3i \notin S$ as 3 is not even. A counter-example of S being an ideal.

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a

$\langle a \rangle = \langle 1 \rangle = \mathbb{Z}$. We know $GCD(2, 3) = 1$ so by *Theorem 0.2* (GCD is a linear combination) (page 4), there are $x, y \in \mathbb{Z}$ such that $2x + 3y = 1$. So for any integer m, $2(xm) + 3(ym) = m$. In other words, $\mathbb{Z} = \langle 1 \rangle \subset \langle 2 \rangle + \langle 3 \rangle$.

b

$\langle a \rangle = \langle 2 \rangle$. Trivially $\langle 6 \rangle + \langle 8 \rangle \subset \langle 2 \rangle$ as 2 is a common divisor of 6 and 8. Observe $8(1) + 6(-1) = 2$. So for any multiple $2m$, We have $8(m) + 6(-m) = 2m$, concluding $\langle 2 \rangle \subset \langle 8 \rangle + \langle 6 \rangle$.

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By definition $A \subset R$ and $r = r1 \in A$ for any $r \in R$.

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Let B be an arbitrary ideal of $\mathbb{Z} \oplus \mathbb{Z}$ such that $A \subset B \subset \mathbb{Z} \oplus \mathbb{Z}$. Assume B properly contains A then we show $B = \mathbb{Z} \oplus \mathbb{Z}$.

By hypothesis we have $(a, b) \in B$ but not in A. So $a = 3q + r$ whereby either $r = 1$ or $r = 2$. Consider each case:

- $r = 1$. Since $A \subset B$, $(3(-q), -(b-1)) \in B$. As B is a group, $(3(-q), -(b-1)) + (3q+1, b) = (1, 1) \in B$.
- $r = 2$. Similarly $(3(q+1), b+1) \in B$ and $(3(q+1), b+1) - (3q+2, b) = (1, 1) \in B$.

By *Ex. 15* $B = \mathbb{Z} \oplus \mathbb{Z}$.

Had A been $\{(4x, y) \mid x, y \in \mathbb{Z}\}$ then the property of it being a maximal ideal fails as the ideal $\{(2x, y)\}$ is strictly larger.

Generally, $\{(rx, y)\}$ is a maximal ideal if and only if r is a prime. If r is composite then any divisor generates a larger ideal. If r is prime then for any m where $0 < m < r$, $gcd(r, m) = 1$. It follows by *Theorem 0.2* (GCD is a linear combination) (page 4) there is a linear combination $xr + ym = 1$.

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If $(x, y), (a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ and $(x, y)(a, b) = (xa, yb) \in I$ then by definition $yb = 0$. So either $y = 0$ or $b = 0$. In other words, either $(x, y) \in I$ or $(a, b) \in I$.

The set $\{(x, 2y) \mid x, y \in Z\}$ is an ideal and properly contains I. So I is not maximal.