Chapter 6 - Section 7

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Contents

Exercises

3

Fact. Given a set A of distinct elements in a random order, The positition of the maximum element of a subset $S \subset A$ is uniform in S.

Define indicator random variables L_i as

$$L_i = \begin{cases} 1 & a_i > a_{i-1}, a_{i-2}, \ldots, a_1 \\ 0 & a_i < a_j, \text{ for some } j = 1, 2, \ldots, i-1 \end{cases}$$

So $L_i = 1$ if and only if the ith item a_i is the maximum in subset $A[1:i]$.

It follows $Pr[L_i = 1] = 1/i$ and $Ex[Li] = 1/i$.

Let X be a random variable for the number of times the line `a[first] > a[max_loc]` returns `True`. Observe $X = L_2 + L_3 + \cdots + L_n$. So $Ex[X] = 1/2 + \cdots + 1/n = H(n) - 1 \approx \ln n - 1$.

$H(n)$ here is the nth harmonic sum.

4

a

Note. Our solution was initially flawed until we read the description of *exercise 6* which gave the correct answer. We only reconstructred the proof given the answer.

Fact 1. On the ith step of the first pass of bubble-sort, $A[i]$ is the maximum element among $A[0:i]$.

Fact 2. Given A is a set of distinct elements in a random order, The probability of $A[i]$ being the maximum element of $A[0:i]$ is $\frac{1}{i+1}$.

Let R_i be an indicator random variable, Indicating whether $A[i] > A[i+1]$, at the ith step of the loop. From *Fact 1*, $R_i = 1$ if and only if $A[i+1]$ is not the maximum among $A[0:i+1]$. The probability of that event is $\frac{i+1}{i+2}$ from *Fact 2*.

Clearly $Ex[R_i] = \frac{i+1}{i+2}$. It follows $W = \sum_{i=0}^{n-2} R_i = \frac{1}{2} + \frac{2}{3} + \cdots + \frac{n-1}{n}$.

b

That event happens if and only if

- $max(A[1], A[2]) < A[3]$. Its probability is $\frac{2}{3}$. Or
- $max(A[1], A[2]) > A[3]$ and $A[1] < A[3]$. Its probability is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

So the probability $A[1] < A[2]$ after the first pass of bubble-sort is $\frac{2}{3} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$.

5

Fact. Given a randomly ordered A, Any $A[: K]$ is also randomly ordered.

Fact. Uniformly $A[k] \in \{q_1, q_2, \ldots, q_k\}$ where $q_i \in A[: k]$ and $q_1 > q_2 > \cdots > q_k$.

In kth iteration, $A[1 : k-1]$ is sorted, and $A[k]$ will be uniformly displaced to

position	$k, k-1, \ldots, 1$. Respectively,
#comparisons	$= 1, 2, \ldots, k$. Respectively,
#assignments	$= 0, 1, \ldots, k-1$.

Denote total number of comparisons by C and comparisons in kth iteration by C_k. Similarly A and A_k for assignments. In expectation

$$Ex[C_k] = \frac{1}{k}(1 + \cdots + k) = \frac{1}{k}\frac{k \cdot k+1}{2} = \frac{k+1}{2}$$
$$Ex[A_k] = \frac{1}{k}(1 + \cdots + k - 1) = \frac{1}{k}\frac{(k-1)k}{2} = \frac{k-1}{2}$$

Clearly $C = \sum_{k=2}^{n} C_k$ and $A = \sum_{k=2}^{n} A_k$. So

$$\begin{aligned}
Ex[C] &= \sum_{k=2}^{n} \frac{k+1}{2} \\
&= \frac{1}{2} \sum_{k=2}^{n} k+1 \\
&= \frac{1}{2} \left[(\sum_{k=1}^{n+1} k) - 1 - 2 \right] \\
&= \frac{1}{2} \left[\frac{(n+1)(n+2)}{2} - 3 \right] \\
&= \frac{(n+1)(n+2)}{4} - \frac{3}{2} \\
Ex[A] &= \sum_{k=2}^{n} \frac{k-1}{2} \\
&= \frac{1}{2} \sum_{k=2}^{n} k-1 \\
&= \frac{1}{2} \sum_{k=1}^{n-1} k \\
&= \frac{1}{2} \frac{n(n-1)}{2} \\
&= \frac{n(n-1)}{4}
\end{aligned}$$